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UICSM Newsletter

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UNIVERSITY OF ILLINOIS COMMITTEE ON SCHOOL MATHEMATICS

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Why a newsletter?

The staff of the University of Illinois Committee on School Mathematics Project feels there is a need for an improved way in which we can communicate with teachers who are using UICSM text materials. We already correspond extensively with the teachers in participating schools who have been to our training conferences, and who send us regular reports on their work, along with queries about content and methods of teaching. However, we have no way of reaching the many people in nonparticipating schools who teach from UICSM materials. We therefore decided a newsletter such as this might be helpful to many people, and that we should begin by sending it to everyone believed by us to be currently teaching from First Course materials.

We have begun by thinking of it as a kind of supplement to, and amplification of, the Teacher's Commentary for our various units. The major criterion to be met by included materials should be their usefulness to the classroom teacher, so we expect to present expository articles relevant to the topics about to be taught in most classes, test items, pedagogical suggestions, additional commentary on those spots in the text apparently most troublesome to students, and notes on the professional activities of UICSM teachers and staff. Perhaps later we can also include anecdotes about notable work by students, letters from teachers, selected references to non-UICSM publications, and maybe even some apt quotations, jokes, and cartoons.

I have asked Alice Hart and Ron Szoke to take the responsibility for the newsletter. Please address contributions and inquiries to either of them in care of the UICSM Newsletter, 1208 W. Springfield, Urbana, Illinois. --Max Beberman, Director, UICSM Project.

PRINCIPAL OPERATOR

Few persons begin difficult undertakings without making arrangements to have the proper tool at hand. In addition to having the tool, one also wishes to have practice in using it in a less complicated situation. In the work on unabbreviating expressions (beginning on page 1-37 of Unit 1) we can introduce a tool that can be used in more complex situations and we can give the students the necessary practice in using it. This tool is the notion of principal operator.

When all the grouping symbols have been restored in unabbreviating an expression the principal operator is the one which corresponds with (or links) the pair of outermost grouping symbols. For example, the principal operator in '$3 + 4 \times 5$' is '+' because unabbreviating this expression gives us:

$$\{3 + (4 \times 5)\}$$

The principal operator in:

$$\{([3 \times 3] + [5 \times 3]) - 10\}$$

is '-'. It may be helpful not to omit the outermost grouping symbols as quickly as we have been doing in the past.

[An expression whose principal operator is '+' is sometimes called 'an indicated sum', one whose principal operator is '×' is called 'an indicated product'.] Before assigning the exercises on page 1-40 we might ask the class to name the principal operator in each of several exercises. A student who is able to do this immediately in an exercise such as:

$$3 + 8 - 2 + 5,$$

$$5 \times 6 \div 3 \times 4,$$

and:

$$4 + [5 - 3 - 1] \times [7 + (2 \times 4)]$$

must be using the conventions correctly.

There are several places in the first four units where the tool, principal operator, may be used to advantage. One of these is discussed in the following pages. During the year, we will bring others to your attention.

* * *

PATTERN-SENTENCES, INSTANCES, CONSEQUENCES

Distinguishing between

> sentences which are instances of a principle
> [and, so, are also consequences of that principle]

and

> sentences which are consequences of a principle
> and are not instances of that principle

is often quite difficult.

We hope that the suggestions which follow will help resolve this difficulty.

Following page 1-45 of Unit 1, arrange to have:

The commutative principle for addition

Instances $\begin{cases} 3+7=7+3 \\ 17+2\frac{1}{2}=2\frac{1}{2}+17 \\ 4.5+2.98=2.98+4.5 \end{cases}$

on one section of your blackboard.

Now begin something like this:

Teacher: Write an instance of the commutative principle for addition on your paper. John, if Mary has actually written an instance of the commutative principle for addition what is one symbol which must be on her paper?

[From here on we will picture the blackboard after the question has been answered.]

$\boxed{\quad = \quad}$ or $\boxed{\quad + \quad}$

Teacher: What other symbols must she have?

$\boxed{\quad = \quad + \quad + \quad}$

[We hope you have to change the above.]

$\boxed{\quad + \quad = \quad + \quad}$

Teacher: What else does she have?

Student: Numerals.

Teacher: Where will one of these numerals be written?

Student: Before the first '+'.

Teacher: Here? [pointing and drawing the '___'.

$\boxed{\;\underline{\quad\quad} + \quad = \quad + \quad\quad}$

Student: A copy of that numeral must be written after the second '+'.

Teacher: Let's indicate that we want a copy of that numeral by using another '___'.

$\boxed{\;\underline{\quad\quad} + \quad = \quad + \underline{\quad\quad}\;}$

Teacher: What else must Mary have?

Student: A numeral after the first '+'.

Teacher: Does that have to be a copy of the numeral we placed before the first '+'? [No!] Let's indicate that by using a '$\sim$'.

$\boxed{\;\underline{\quad\quad} + \sim = + \;\underline{\quad\quad}\;}$

[Continue until you have a pattern-sentence]

$$\underline{\quad\quad} + \sim\sim = \sim\sim + \underline{\quad\quad}$$

Teacher: We call this a pattern-sentence for the commutative principle for addition. If you have actually written an instance of the commutative principle for addition, then if we put the proper numerals in the blanks in this pattern-sentence we will have a copy of the sentence you wrote.

Mary, tell me how to fill in these blanks so that we will have a copy of your sentence.

Student: Put a '4' above each '$\underline{\quad\quad}$' and a '7' above each '$\sim\sim$'.

Teacher: Is this a copy of your sentence?

Student: Yes.

$$\underset{\underline{\quad\quad}}{4} + \underset{\sim\sim}{7} = \underset{\sim\sim}{7} + \underset{\underline{\quad\quad}}{4}$$

Teacher: Then your sentence is an instance of the commutative principle for addition. Look at this sentence. Is this an instance of the commutative principle for addition?

$$\underline{\quad\quad} + \sim\sim = \sim\sim + \underline{\quad\quad}$$
$$5+7+3 = 7+5+3$$

Let's unabbreviate it.

$$\underline{\quad\quad} + \sim\sim = \sim\sim + \underline{\quad\quad}$$
$$\{(5+7)+3\} = \{(7+5)+3\}$$

What is the principal operator on the left side of the pattern sentence? What is the principal operator on the left side of the other sentence?

$\underline{\qquad} + \sim\!\sim\!\sim = \sim\!\sim\!\sim + \underline{\qquad}$	$\{(5+7)+3\} = \{(7+5)+3\}$

Teacher: If this is an instance, then we must be able to fill the blanks in the pattern sentence so that we get a copy. What shall we place above the '$\sim\!\sim\!\sim$' on the left side of the pattern-sentence?

Student: A '3'.

Teacher: Where else must a '3' go?

Student: Above the '$\sim\!\sim\!\sim$' on the right.

Teacher: What must go above the '___' on the left side?

Student: A '(5 + 7)' above each '___'.

$$\underline{(5+7)} + \underset{\sim\sim\sim}{3} = \underset{\sim\sim\sim}{3} + \underline{(5+7)}$$

Teacher: Is this a copy of our original sentence?

Student: No.

Teacher: Then our original sentence is <u>not</u> an instance of the commutative principle of addition. However, an instance of the commutative principle for addition can be used to help us decide if such a sentence as:

$$(759 + 78) + 95 = (78 + 759) + 95$$

is true.

Show: $(759+78)+95=(78+759)+95$

$$\underline{\qquad} + \sim\!\sim\!\sim = \sim\!\sim\!\sim + \underline{\qquad}$$

What is the instance?

Student: '$\underline{759} + \underset{\sim\sim\sim}{78} = \underset{\sim\sim\sim}{78} + \underline{759}$'.

Show: $(759 + 78) + 95 = (78 + 759) + 95$

$(759 + 78) = (78 + 759)$

Teacher: If we accept the commutative principle for addition then without doing any computation we must believe that '759 + 78' and '78 + 759' name the same number.

Show: $(759 + 78) + 95 = (78 + 759) + 95$

$(759 + 78) = (78 + 759)$ [cpa]

$(759 + 78) + 95 =$ ________

What is the simplest way to complete this sentence so that we will have a true statement?

Student: '95 + (759 + 78)'.

Teacher: Too hard.

Student: '932'.

Teacher: Too hard.

Student: '(759 + 78) + 95'.

Teacher: Right!!!

Show: $(759 + 78) + 95 = (78 + 759) + 95$

(*) $(759 + 78) = (78 + 759)$ [cpa]

$(759 + 78) + 95 = (759 + 78) + 95$

Teacher: Now how are we going to use (*) to show that the sentence at the top of the board is true?

[Deathly silence!!! But <u>wait</u>. If it doesn't come, go ahead.]

What does (*) tell you? (Mean to you?)

Student: (*) tells me that '(759 + 78)' and '(78 + 759)' are names for the same number.

Teacher: Do you <u>have</u> to use the name '(759 + 78)'?

Student: No.

Teacher: Do you <u>have</u> to use the name '(78 + 759)'?

Student: No.

Teacher: Can you use either name you decide you want to use?

Student: Yes.

Teacher: Let's look at:

$$(759 + 78) + 95 = (759 + 78) + 95$$

and decide whether we want to use the name '(759 + 78)' or the name '(78 + 759)'. Which one do you want to use?

Student: Keep a '(759 + 78)' on the left side and use a '(78 + 759)' on the right side.

Teacher: What shall I write?

$$\boxed{\begin{array}{l} \underline{\text{Show: } (759+78)+95=(78+759)+95} \\ \quad (759+78)=(78+759) \quad [\text{cpa}] \\ (759+78)+95=(\underline{759+78})+95 \\ (759+78)+95=(78+759)+95 \end{array}}$$

Is this what we wanted?

Student: Yes

Teacher: We usually abbreviate all this as:

$$\boxed{(759+78)+95=(78+759)+95 \quad [\text{cpa}]}$$

Is this sentence an instance of the cpa?

Student: No.

Teacher: Did we use an instance of the cpa?

Student: Yes.

Teacher: We say that

$$(759 + 78) = (78 + 759)$$

is <u>both</u> an instance <u>and</u> a consequence of the cpa.

Teacher: However,

$$(759 + 78) + 95 = (78 + 759) + 95$$

is a consequence but is <u>not</u> an instance of the cpa.

[Obviously, the dialogue above is intended only to indicate possible questions and answers.]

*

Now develop pattern sentences for the cpm, the apa, and the apm.

cpm: $\_\_\_ \times \sim\sim\sim = \sim\sim\sim \times \_\_\_$

apa: $(\_\_\_ + \sim\sim\sim) + ---- = \_\_\_ + (\sim\sim\sim + ----)$

or $\_\_\_ + (\sim\sim\sim + ----) = (\_\_\_ + \sim\sim\sim) + ----$

or $\_\_\_ + \sim\sim\sim + ---- = \_\_\_ + (\sim\sim\sim + ----)$

or $\_\_\_ + (\sim\sim\sim + ----) = \_\_\_ + \sim\sim\sim + ----$

apm: similar to those for the apa.

[Be certain that the students see that the four pattern-sentences for the apa are equivalent.]

* * *

[The following material is very much condensed.]

Teacher: Now let's examine Exercise 14 of A on page 1-48:

$$72 + (45 + 63) + 85 = 85 + [72 + (45 + 63)]$$

We are told this is an instance of one of the four principles. First, let's unabbreviate.

$$\{[72 + (45 + 63)] \underset{\uparrow}{+} 85\} = \{85 \underset{\uparrow}{+} [72 + (45 + 63)]\}$$

What is the principal operator in the expression on the left side?

Student: The third '+'.

Teacher: In the expression on the right side?

Student: The fourth '+'. [The first '+' on the right side.]

Teacher: These '+' signs must match the principal operators in the two sides of one of the patterns. Which pattern do you want to try?

Student: $$\_\_\_ \underset{\uparrow}{+} \underset{\sim\sim\sim}{85} = \underset{\sim\sim\sim}{85} + \_\_\_$$

$$[\underline{72 + (45 + 63)}] + 85 = 85 + [\underline{72 + (45 + 63)}]$$

Teacher: Is this a copy of the unabbreviated sentence?

Student: Yes.

Teacher: Then the original sentence is an instance of the cpa.

Now look at Exercise 13, and unabbreviate. What is the principal operator on the first side? On the second side?

$$\{(72 + 45) \underset{\uparrow}{+} (63 + 85)\} = \{72 \underset{\uparrow}{+} [45 + (63 + 85)]\}$$

$$\underline{(72 + 45)} + (63 + 85) = (63 + 85) + \underline{(72 + 45)}$$

Teacher: Is this a copy?

Student: No.

Teacher:

$$\_\_\_ \underset{\uparrow}{+} (\sim\sim + ----) = (\_\_\_ + \sim\sim) + ----$$

$$\underline{(72 + 45)} \underset{\uparrow}{+} (\sim\sim + ----) = \left(\underline{(72 + 45)} + \sim\sim\right) \underset{\uparrow}{+} ----$$

$$\underline{(72 + 45)} + (63 + 85) = \left(\underline{(72 + 45)} + 63\right) + 85$$

Is this a copy?

Student: No.

Teacher: Let's see if the second side of our sentence will fit the first side of the pattern:

$$\underline{72} + (\sim\sim + ----) = (\underline{72} + \sim\sim) + ----$$

$$\underline{72} + \left(45 + (63 + 85)\right) = (\underline{72} + \sim\sim) + (63 + 85)$$

Is this a copy of our sentence?

Student: Not quite.

[Now we can emphasize that if we believe that '72 + [45 + (63 + 85)]' and '(72 + 45) + (63 + 85)' name the same number, then we should believe that '(72 + 45) + (63 + 85)' and '72 + [45 + (63 + 85)]' name the same number. Paying attention to the symmetric property of equality at this time will help in the proofs later.]

Teacher: So the sentence in Exercise 13 is an instance of the apa.

We hope that this development using the tool, principal operator, will be helpful in the discussion of instances and consequences.

UICSM-NETRC Tests

A series of 18 short tests on Units 1, 2, and 3 has been prepared for use in the teacher-training experiment sponsored by UICSM and the National Educational Television and Radio Center. All are timed and of the "objective" type. Most are a little speeded. We present here the first four of these, though they probably cannot be used until next year by most teachers. The tests are lettered serially from A through R and the number of the page of the text which should have been covered before the test is administered is given in brackets immediately after the letter. UICSM teachers should feel free to use any or all of these items in their tests and quizzes and to send us improved versions of any that seem unclear or otherwise defective. The Newsletter staff will of course be pleased to receive any new and ingenious items devised by teachers who wouldn't mind sharing them with other UICSM teachers. --R.S.

Test A [1-7] (10 minutes)

Directions: Mark 'A' if the sentence is true. Mark 'B' if it is not true.

1. Most people would be frightened if they found 'a live cobra' in their desks.

2. 'Alaska' = 'the largest state in the United States'

3. George Washington = the first president of the United States

4. '(13 + 3) ÷ 4' is a name for '6 - 2'.

5. The word 'ambiguous' is printed below:

   ambiguous

6. '3' is not a number.

7. 17 - 3 is the same as 2 × 7.

8. If told to write a numeral for 5, you would be correct if you wrote:

   '4 + 1'

9. 'Chicago' is a large city.

10. '10 - 3' is a numeral for 7.

11. '$(3 + 5) \times \frac{1}{2}$' is larger than '7'.

12. 0 is a numeral having an oval shape.

Assume a single straight "road" for the trips in problems 13 through 24.

13. If the distance from A to B is twice the distance from B to C, then C is farther from A than B is.

14. If a man takes a trip of 20 miles, then another trip of 20 miles, he will then be 40 miles from his original starting point.

15. If two people each take a trip measured by the same real number, they will then be just as far apart as they were when they started.

16. If two people each take a trip of one mile, they will then be just as far apart as they were when they started.

17. If C is north of A and also north of B, then B is north of A.

18. If the trip from A to B is measured by the same real number as the trip from C to D, then the trip from A to C is measured by the same real number as the trip from B to D.

19. If the result of several successive directed trips is a return to the original starting point, some of the trips were in opposite directions.

20. Numbers of arithmetic are not used to measure trips because they do not tell how long each trip is.

21. After the unit and positive direction have been chosen, each possible trip is measured by just one real number.

22. After the unit and positive direction have been chosen, each real number measures just one trip.

23. If the trip from M to L is measured by the same real number as the trip from N to L, then N is twice as far from M as L is.

24. Suppose you are to take a trip from the starting point of the first to the ending point of the last of the successive trips measured by some given real numbers. Then no matter which direction was chosen as negative, this trip will be measured by the same real number.

Key for Test A [1-7]:

1. B	2. B	3. A	4. B	5. A	6. A
7. A	8. B	9. B	10. A	11. A	12. B
13. B	14. B	15. A	16. B	17. B	18. A
19. A	20. B	21. A	22. B	23. B	24. A

Test B [1-15] (15 minutes)

I. Choose the correct simplification:

1. $^-13 + {}^+3$

(A) $^+10$ (B) $^-10$ (C) $^+16$ (D) $^-16$

2. $^+6 + {}^-8$

(A) $^+14$ (B) $^-14$ (C) $^+2$ (D) $^-2$

3. $({}^{-}2 + {}^{+}7) + {}^{-}5$

(A) ${}^{+}14$ (B) 0 (C) ${}^{+}4$ (D) ${}^{-}4$

4. $({}^{+}2 + {}^{-}7) + {}^{-}9$

(A) ${}^{-}4$ (B) 0 (C) ${}^{-}16$ (D) ${}^{-}14$

5. $[({}^{+}13 + {}^{-}5) + {}^{+}5] + {}^{-}13$

(A) ${}^{+}18$ (B) ${}^{+}8$ (C) ${}^{-}8$ (D) 0

6. ${}^{-}29.03 + {}^{+}99.98$

(A) ${}^{+}70.95$ (B) ${}^{-}70.95$ (C) ${}^{+}129.01$ (D) ${}^{-}129.01$

7. $({}^{-}37 + {}^{+}25) + {}^{-}12$

(A) ${}^{-}24$ (B) ${}^{+}24$ (C) ${}^{-}74$ (D) 0

8. $({}^{+}231.57 + {}^{-}548.59) + {}^{+}18.37$

(A) ${}^{-}335.39$ (B) ${}^{-}298.65$ (C) ${}^{-}266.18$ (D) ${}^{-}213.20$

II. Choose the answer which will make the sentence true:

9. $\_\_\_\_\_ + {}^{-}3 = {}^{-}20$

(A) ${}^{+}23$ (B) ${}^{-}23$ (C) ${}^{-}17$ (D) ${}^{+}17$

10. ${}^{-}14.8 + {}^{+}6.9 = \_\_\_\_\_ + {}^{-}7.9$

(A) ${}^{+}13.8$ (B) 0 (C) ${}^{+}15.8$ (D) ${}^{+}1$

11. $({}^{+}3 + \_\_\_\_\_) + {}^{-}5 = {}^{-}3$

(A) ${}^{+}5$ (B) ${}^{+}8$ (C) ${}^{-}2$ (D) ${}^{-}1$

12. $\_\_\_\_\_ + {}^{-}3 = \_\_\_\_\_ + {}^{-}(24 \div 6)$

(A) '4' in both blanks (B) '${}^{-}4$' in both blanks

(C) '0' in both blanks (D) none of these

13. $({}^{+}7800 + {}^{-}85) + \_\_\_\_\_ = {}^{+}85$

(A) ${}^{-}7630$ (B) ${}^{-}7715$ (C) ${}^{-}7800$ (D) ${}^{-}7885$

14. ${}^{-}(9 - 3) = \_\_\_\_\_ + {}^{-}3$

(A) ${}^{-}3$ (B) ${}^{-}6$ (C) 9 (D) 12

15. ${}^{-}[(10 - 7) + 15] = ({}^{-}10 + {}^{-}7) + \_\_\_\_\_$

(A) ${}^{-}1$ (B) ${}^{+}15$ (C) ${}^{-}15$ (D) ${}^{+}35$

16. $^{-}(27 - 3) +$ _____ $= ($_____ $+ {}^{-}37) + {}^{+}25$

    (A) '0' in the first blank and '$^{+}12$' in the second

    (B) '$^{+}4$' in the first blank and '$^{-}8$' in the second

    (C) '$^{-}12$' in both blanks

    (D) none of these

III. Choose the correct answer to the question:

17. What is the sum of a negative number and a negative number?

    (A) a positive number (B) a negative number

    (C) 0 (D) cannot tell

18. What is the sum of a positive number and a negative number?

    (A) a positive number (B) a negative number

    (C) 0 (D) cannot tell

19. If the sum of a positive number and another number is a negative number, what was added to the positive number?

    (A) a positive number (B) a negative number

    (C) 0 (D) cannot tell

20. If the sum of a negative number and another number is zero, what was added to the negative number?

    (A) a positive number (B) a negative number

    (C) 0 (D) cannot tell

Key for Test B [1-15]:

1. B	2. D	3. B	4. D	5. D	6. A	7. A
8. B	9. C	10. B	11. D	12. D	13. A	14. A
15. A	16. B	17. B	18. D	19. B	20. A	

Test C [1-32] (15 minutes)

Be careful not to confuse addition signs with multiplications signs.

I. Choose the correct simplification:

1. $6 \times {}^{-}8$

    (A) 48 (B) $^{-}48$ (C) $^{-}2$ (D) 14

2. $^{-}3 \times 5$

    (A) 2 (B) 15 (C) $^{-}15$ (D) $^{-}8$

3. ${}^{-}7 \times {}^{-}8$

(A) ${}^{-}15$ (B) ${}^{-}56$ (C) 56 (D) ${}^{-}1$

4. $(3 + {}^{-}5) \times 6$

(A) 4 (B) 12 (C) ${}^{-}12$ (D) ${}^{-}90$

5. $({}^{-}5 \times {}^{-}6) + {}^{-}30$

(A) ${}^{-}900$ (B) ${}^{-}60$ (C) 0 (D) 60

6. $18 + ({}^{-}6 \times {}^{-}3)$

(A) 36 (B) 9 (C) 324 (D) 0

7. $({}^{-}13 \times 4) \times {}^{-}(\frac{1}{13})$

(A) $\frac{9}{13}$ (B) 4 (C) ${}^{-}4$ (D) ${}^{-}(\frac{52}{13})$

8. $[({}^{-}3 \times 2) + ({}^{-}1 \times {}^{-}6)] \times {}^{-}9$

(A) 0 (B) 9 (C) 108 (D) ${}^{-}108$

II. Choose the answer which will make the sentence true:

9. ${}^{-}7 \times$ _____ $= {}^{-}56$

(A) 63 (B) ${}^{-}49$ (C) ${}^{-}8$ (D) 8

10. ${}^{-}14 \times$ _____ $= (7 + {}^{-}10) + ({}^{-}3 \times {}^{-}1)$

(A) $\frac{3}{7}$ (B) ${}^{-}(\frac{3}{7})$ (C) $\frac{1}{2}$ (D) 0

11. ${}^{-}4 \times$ _____ $= 1$

(A) ${}^{-}.25$ (B) $\frac{1}{4}$ (C) 4 (D) 5

12. $4 \times$ _____ $= {}^{-}6 \times 8$

(A) 16 (B) ${}^{-}12$ (C) 2 (D) ${}^{-}2$

13. $(5 +$ _____ $) \times {}^{-}3 = 3$

(A) ${}^{-}6$ (B) ${}^{-}4$ (C) ${}^{-}(\frac{1}{5})$ (D) $\frac{6}{5}$

14. $(2 \times$ _____ $) + 1 = {}^{-}9$

(A) ${}^{-}12$ (B) ${}^{-}5$ (C) ${}^{-}4$ (D) 4

15. $2 \times$ _____ $= 3 \times$ _____

(A) '0' in each blank (B) '6' in each blank

(C) '${}^{-}6$' in each blank (D) none of these

16. (3 × ______) + 2 = ______ + (2 × ______)

(A) '1' in each blank (B) '$^{-}1$' in each blank

(C) '0' in each blank (D) none of these

III. Choose the correct answer to the question:

17. What is the product of a positive number and a negative number?

(A) a positive number (B) a negative number

(C) 0 (D) cannot tell

18. If the product of zero and a real number is zero, by what was zero multiplied?

(A) a positive number (B) a negative number

(C) 0 (D) cannot tell

19. If the product of 1 and a real number is the real number, by what was 1 multiplied?

(A) a positive number (B) a negative number

(C) 0 (D) cannot tell

20. If the product of a real number and a real number is zero, what can you conclude about the real numbers?

(A) both are nonnegative (B) both are zero

(C) at least one of them is zero (D) nothing

Key for Test C [1-32]:

1. B	2. C	3. C	4. C	5. C	6. A	7. B
8. A	9. D	10. D	11. A	12. B	13. A	14. B
15. A	16. D	17. B	18. D	19. D	20. C	

Test D [1-59] (15 minutes)

Only numbers of arithmetic are used in this test.

I. Choose the answer that will make the sentence true.

1. 7 × 2 + 7 × ______ = (2 + 3) × 7

(A) 0 (B) 2 (C) 3 (D) 5

2. $5 \times 3\frac{1}{3} = \_\_\_\_\_\_ + \frac{5}{3}$

(A) 3 (B) 5 (C) 10 (D) 15

3. $7 \times 8 + 5 \times 8 = 40 + \_\_\_\_ \times 8$

(A) 5 (B) 6 (C) 7 (D) 8

4. $7 \times 13 + 13 \times \_\_\_\_ = 13 \times 13$

(A) 13 (B) 10 (C) 7 (D) 6

5. $(5 + \_\_\_\_) \times 5 = 25 + 40$

(A) 3 (B) 5 (C) 8 (D) 11

6. $13 \times \_\_\_\_ + \_\_\_\_ \times 7 = 100$

(A) '4' in both blanks (B) '5' in both blanks

(C) '6' in both blanks (D) none of these

7. $987 \times 593 + 13 \times 593 = \_\_\_\_$

(A) 98,700 (B) 58,700 (C) 59,300 (D) 593,000

8. $4 \times 3 + 4 \times \_\_\_\_ = 4 \times \_\_\_\_ + 15$

(A) '3' in both blanks (B) '4' in both blanks

(C) '5' in both blanks (D) none of these

9. $13 + 2 \times 11 = 13 \times 2 + \_\_\_\_$

(A) 7 (B) 9 (C) 11 (D) 143

10. $139 \times 672 + 139 \times 328 = \_\_\_\_$

(A) 139,000 (B) 13,900 (C) 46,700 (D) 146,700

II. Choose the principle which justifies the given sentence.

11. $7 + 2 \times 3 = 7 + 3 \times 2$

(A) apm (B) cpa (C) cpm (D) apa (E) dpma

12. $0 = 19 \times 0$

(A) apm (B) cpm (C) pm0 (D) pa0 (E) ℓdpma

13. $5 + 3 + 8 = 5 + (3 + 8)$

(A) cpa (B) cpm (C) apa (D) apm (E) dpma

14. $(47 + 3) \times 9 = [(47 + 3) + 0] \times 9$

(A) cpm (B) apm (C) apa (D) pm0 (E) pa0

15. $(3 + 14) \times 7 = (14 + 3) \times 7$
(A) cpm (B) cpa (C) apa (D) dpma (E) ℓdpma

16. $4 \times (3 + 19) = 4 \times 3 + 4 \times 19$
(A) apm (B) dpma (C) ℓdpma (D) cpm (E) apa

17. $(48 + 19) \times 1 + 12 = 48 + 19 + 12$
(A) dpma (B) ℓdpma (C) apm (D) pml (E) apa

18. $37 \times 1 \times (12 - 3) \times 7 = 37 \times 1 \times [(12 - 3) \times 7]$
(A) apm (B) cpm (C) apa (D) cpa (E) pml

19. $5 \times 1\frac{3}{8} = 5 \times 1 + 5 \times \frac{3}{8}$
(A) pml (B) apm (C) cpm (D) ℓdpma (E) dpma

20. $(\frac{1}{3} + \frac{1}{2}) \times 6 = \frac{1}{3} \times 6 + \frac{1}{2} \times 6$
(A) dpma (B) ℓdpma (C) apa (D) apm (E) pml

III. Choose the correct answer.

21. Which of the following is an <u>instance</u> of apm?
(A) $(72 \times 45 \times 63) \times (85 \times 22) = (72 \times 45) \times [63 \times (85 \times 22)]$
(B) $71 \times (52 \times 13) = (52 \times 13) \times 71$ (C) $5 \times 3 \times 4 = (5 \times 3) \times 4$
(D) $23 \times (5 \times 1) = 23 \times 1 \times 5$ (E) none of them

22. Which of the following is a consequence of <u>just</u> apa?
(A) $2 + 3 \times 4 = 2 \times (3 + 4)$
(B) $5 + 7 + 2 = 7 + (5 + 2)$
(C) $[(7 + 4) + 6] + 8 = (7 + 4) + (6 + 8)$
(D) $6 + (8 + 9) = (6 + 9) + 8$
(E) none of them

23. Which of the following is a consequence of our punctuation convention alone?
(A) $72 + (45 + 63) + 85 = 72 + (45 + 63 + 85)$
(B) $(72 + 45) + (63 + 85) = 72 + 45 + (63 + 85)$
(C) $(72 + 45 + 63) + 85 = (72 + 45) + (63 + 85)$
(D) $72 + 45 + (63 + 85) = 72 + [45 + (63 + 85)]$
(E) none of them

24. Which of the following is an <u>instance</u> of cpm?

(A) $8 \times 3 + 7 = 3 \times 8 + 7$

(B) $5 \times (4 + 3) = (3 + 4) \times 5$

(C) $2 \times (1 + 3) = 2 \times 1 + 3 \times 2$

(D) $13 \times 5 + 8 \times 5 = 5 \times 13 + 5 \times 8$

(E) none of them

Key for Test D [1-59]:

1. C	2. D	3. C	4. D	5. C	6. B
7. D	8. D	9. B	10. A	11. C	12. C
13. C	14. E	15. B	16. C	17. D	18. A
19. D	20. A	21. A	22. C	23. B	24. E

* * *

News and Notices

PLEASE

send us your correct address, or the address to which you would prefer future issues of this Newsletter be sent. We will also be grateful for the names and addresses of others whom you feel should receive copies of this publication. A postal card will suffice.

UICSM Newsletter
1208 W. Springfield
Urbana, Illinois

Mr. Beberman will be traveling extensively during the next three months while visiting cooperating schools and making presentations before various professional groups. His itinerary follows.

October

10-13 Visit Pine Bluff, Arkansas, schools.

14-15 Presentation to Arkansas Council of Teachers of Mathematics, Arkansas State College, Conway.

21 Presentation at Western Zone meeting of the mathematics section, New York State Teacher's Association, Snyder.

24-25 Visiting Villa Maria Academy and Cathedral Preparatory School, Erie, Pennsylvania.

27 Presentation to meeting of School Superintendent's Division, Catholic Educational Association, Peoria, Illinois.

28-29 Mathematics Teacher's Symposium, Villanova University, Villanova, Pa.

31 November 1 — Visit Cheltenham, Pa., public schools.

2 Visit Ridgewood, N.J., public schools.

3 Visit Pascack Valley Regional High School, Hillsdale, N.J.

10 Visit public schools in Cleveland Heights, Cleveland, Ohio.

14-16 Visit Pittsburgh, Pa., cooperating schools.

17-18 Presentation at annual meeting of Edison Foundation, Pittsburgh.

December

8 Visit Whittier, California, public schools.

9-11 Presentation to annual meeting of California mathematics teachers at Asilomar.

13 Presentation to National Science Foundation Institute at Sacramento State College, Sacramento, California.

30 Presentation at Christmas meeting of the National Council of Teachers of Mathematics, Tempe, Arizona.

Mr. Eugene Epperson of Talawanda High School, Oxford, Ohio, will speak at the annual meeting of the Western Ohio Teacher's Association in Dayton on October 28. He is also to appear as a teacher-panelist at the Regional Orientation Conference in Cincinnati on December 15th.

Miss Gertrude Hendrix, Project Teacher Coordinator and Programmer for NETRC Math Study films, is the author of "The Case for Basic Research in the Theory of Instruction" in the May, 1960, American Mathematical Monthly (vol. 67, pp. 446-7). She also co-authored "The UICSM Teacher Training Films" in the August-September, 1960, issue of the same journal (vol. 67, pp 686-7) with Mr. Byrl Sims, who directed the film unit.

Five former UICSM students are among the seven from the new Newton (Mass.) South High School who passed the qualifying examination of the National Merit Scholarship Program. Their names are Ray Frieden, Ralph Pollack, Shepard Golub, Philip Alpert, and Laura Cohen.

Reprints of Mr. Beberman's article on "Improving High School Mathematics Teaching", as it appeared in the December, 1959, Educational Leadership, are still available from the project office. The following reprints are also still available:

UICSM Project Staff. Words, 'Words', "Words".

UICSM Project Staff. Arithmetic with Frames.

Gertrude Hendrix. Variable Paradox: A Dialog in One Act.

M. Eleanor McCoy. A Secondary School Mathematics Program.

UICSM Newsletter

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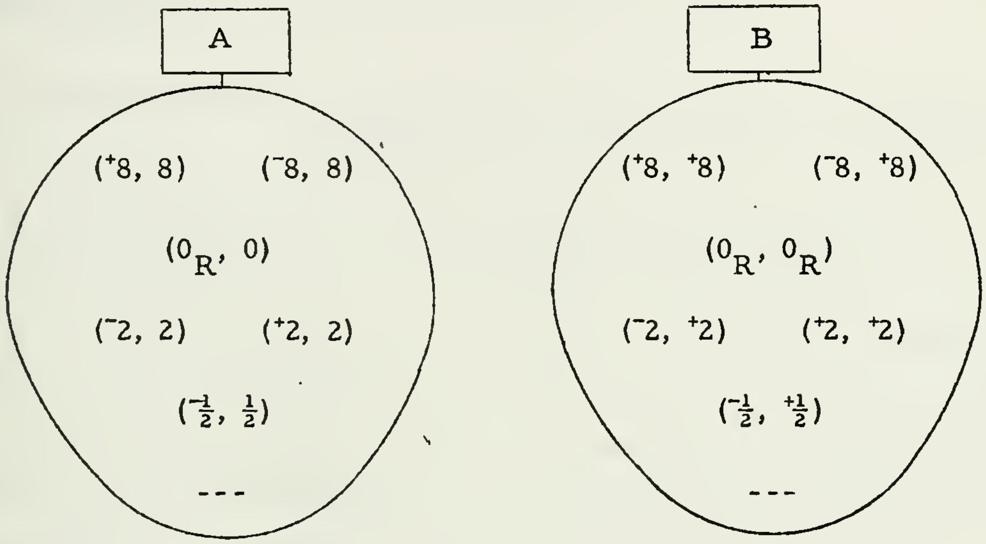
UICSM Newsletter No. 2 November 9, 1960

ABSOLUTE VALUING: TWO OPERATIONS, ONE NAME

Many of the questions asked about material in Course I are inquiries about absolute valuing. These questions are closely related to the problem raised by using numerals for numbers of arithmetic as names for real numbers. This problem would be simplified by a certain amount of rewriting in Unit I. If you are interested in these contemplated changes, mail us a card and we'll send you a copy of them.

Until changes are made in the text, perhaps some of the confusion concerning absolute valuing can be eliminated by the following discussion:

Consider the operations A and B, some of whose ordered pairs are named below:



[In these tables numerals for numbers of arithmetic are not being used as names for real numbers.]

A is an operation which maps the reals onto the set of numbers of arithmetic. B is an operation which maps the reals onto the set of nonnegative reals. Since these operations contain different ordered pairs, $A \neq B$.

Operations A and B are both important. A and its "partial inverses", $^{+}$ and $^{-}$, are needed in order to get back and forth between the real numbers and the numbers of arithmetic--for each real number x, A(x) is the number of arithmetic "corresponding" to x, and, for each number a of arithmetic, ${}^{+}a$ and ${}^{-}a$ are, respectively, the nonnegative real number and the nonpositive real number which "correspond" to a. Thus, for example, these three operations are used when one adds or multiplies

real numbers by adding, subtracting, or multiplying numbers of arithmetic. And, notation for them is needed if, as on pages 2-28 and 2-29, one is to state the rules for adding and multiplying real numbers. Another place where we make explicit use of A [and use '| |' as a name for A] is in Unit 6. We do so when discussing coordinate systems on pages 6-232 and 6-233; in exercises, on pages 6-235, 6-236, and 6-237, leading up to the distance formula; and in the distance formula itself, on pages 6-238 and 6-239.

The isomorphism between the system of nonnegative real numbers and the system of numbers of arithmetic is expressed by:

$$\forall_x \text{ if } x \text{ is nonnegative then } {}^{+}A(x) = x, \quad \forall_x\ A({}^{+}a) = a$$
$$[\text{domain of 'a'} = \text{set of numbers of arithmetic}],$$

$$\forall_x \forall_y \text{ if } x \text{ and } y \text{ are nonnegative then } x + y = {}^{+}(A(x) + A(y))$$
$$\text{and } x \cdot y = {}^{+}(A(x) \cdot A(y)),$$

$$\forall_x \forall_y \text{ if } x \text{ and } y \text{ are nonnegative then } x > y \text{ if and only if } A(x) > A(y)$$

It is this isomorphism which suggests that numerals which have been introduced as names for numbers of arithmetic may, on occasion, be used as names for nonnegative real numbers [see page 1-31]. [In anticipation of this we have, from the beginning, used the symbols '+', 'X', and '>', which originally refer to two operations and a relation for numbers of arithmetic, to refer to the "corresponding" operations and relation for real numbers.] It is also the existence of this isomorphism which makes it possible for us to take a further step and "pretend that measures [which are numbers of arithmetic] are real numbers" ([6-35]) and to "think of them "as if they were real numbers" " (TC[3-55]). See, also, TC[5-196].

The operation A is, as indicated above, important in two kinds of situations. First, when one is dealing with the foundations of the arithmetic of real numbers as based on the arithmetic of the numbers of arithmetic, and, second, when one is applying theorems proved for real numbers to solve problems which are, actually, concerned with numbers of arithmetic.

The second kind of situation occurs very frequently--in particular in the solution of worded problems in Units 3, 4, and 5, and throughout Unit 6. However, explicit reference to A and $^+$ is seldom made in this connection. But see, again, TC[5-196] and the references there cited.

The operation B is of importance in the theory of real numbers itself. It can be introduced without reference to numbers of arithmetic either by:

$\forall_x$ if x is nonnegative then $B(x) = x$ and if x is nonpositive then $B(x) = -x$

or, more succinctly, by:

$\forall_x \; B(x) = \sqrt{x^2}$

Alternatively, B may be defined in terms of A and $^+$:

$\forall_x \; B(x) = {}^+A(x)$ [For, $\forall_x$ if x is nonpositive then ${}^+A(x) = -x$.]

The most frequent use we make of B is to obtain a wider range of exercises than we otherwise could. There is not much of this in Unit 2. But see, especially, the last sentence beginning on TC[2-3, 4, 5]a--this should be recalled when doing Part B on page 2-111. There are many examples of this use of B in Unit 3. Also, our only two "professional" uses of B are in Unit 3, pages 3-118 et seq., and page 3-132 et seq.

Of the three possible definitions of B, noted above, only the first and third are pedagogically suitable. Because we have operation A and operation $^+$ available, and for another reason, we chose to use the third. The other reason is that we follow the custom of using numerals for numbers of arithmetic <u>sometimes</u> to name numbers of arithmetic and <u>sometimes</u> to name the corresponding nonnegative real numbers. [This is an unfortunate custom, but it is so wide-spread that students must learn to live with it.] Because of this custom, a '2', in a given context, may be intended to name the number 2 of arithmetic <u>or</u> it may be intended to name the real number $^+2$. [Of course, only a schizophrenic can use the same '2' to name <u>both</u> the number 2 of arithmetic and the real number $^+2$.] For example, in the expression '$^+2$', the '2' must be intended as a name for a number of arithmetic, for the domain of the operation $^+$ contains only such numbers; and in the expressions 'A(2)'

and 'B(2)' the '2's can only be names for $^+2$, for the domain of either A or B contains only real numbers. Now, one consequence of adopting this custom is that, having introduced '$|\ldots|$' as a substitute for 'A(. . .)', the expression '$|^+2|$' [or '$|2|$'] which, at the time '$|\ldots|$' is introduced, is a name for the number 2 of arithmetic, may, in the future, be used sometimes as a name for this number and sometimes as a name for $^+2$. In the latter case, '$|\ldots|$' is being used as a substitute for 'B(. . .)', instead of for 'A(. . .)'. So, following the custom means, in this connection, that the same symbol '$|\ |$' will refer sometimes to the operation A and sometimes to the operation B. A somewhat similar situation has already occured in connection with '+', '×', and '<'. For example, '+' refers sometimes to the operation of addition of numbers of arithmetic and sometimes to the operation of addition of real numbers. The difference in the two situations is that in, for example, '2 + 3', one knows which operation '+' refers to as soon as one has learned [from the over-all context] which kind of numbers '2' and '3' are being used as names for. But, in the case of '$|2|$' one knows that, in any case, the '2' is being used as a name for $^+2$, but one has to look at the over-all context to determine which operation the '$|\ |$' refers to.

Now, ever since the custom we have been discussing was first introduced, on pages 1-31 and 1-32, we have acted according to the convention that numerals which may be used to name numbers of arithmetic will, in fact, be interpreted as naming the corresponding nonnegative real numbers whenever the context does not, either explicitly or implicitly, prohibit this interpretation. So, from Unit 2 on, '$|\ |$' refers to the operation B unless the context prohibits this interpretation. Thus, the sentence:

$$|^-5| > 2$$

is to be considered as a sentence about real numbers and is equivalent to:

$$^+5 > {}^+2$$

However, the sentence:

$$^-5 + {}^+2 = {}^-(|^-5| - |^+2|)$$

is meaningless if we think of '$|^-5|$' as a name for $^+5$ since nonpositiving is an operation on numbers of arithmetic. So here we must think of

'$|^{-}5|$' and '$|^{+}2|$' as names for numbers of arithmetic.

For another example, in Exercise 9, on page 2-4, the context does not prohibit [it even suggests] the interpretation of '$|\ |$' as B. So, this is the interpretation to be given to '$|\ |$' in this exercise. On the other hand, on page 2-29, the context, which includes the last three lines on page 2-28, implicitly prohibits the interpretation of the two occurrences of '$|\ |$' on line 5 as references to B [and the phrase 'which corresponds with' might be interpreted as explicitly prohibiting this interpretation]. Also, the '$^{-}$' in line 12 explicitly prohibits the interpretation of the two '$|\ |$'s on this line as references to B. So, all four of these occurrences of '$|\ |$' must be taken as referring to A. In D, 1(a), on page 2-29, the last two '$|\ |$'s must, for the same reason, refer to A. There may be some doubt about the first two. Certainly, either both refer to A or both refer to B. Because of the isomorphism, with respect to >, between the numbers of arithmetic and the nonnegative real numbers, the truth or falsity of a statement of the form '$|x| \geq |y|$' is unaffected by the choice of either interpretation. However, the over-all context--defining the arithmetic of the real numbers in terms of that of the numbers of arithmetic -would probably be taken as prohibiting the interpretation of these '$|\ |$'s as referring to B.

For another example, in Unit 6, on page 6-232, since [whatever it may be convenient to pretend] measures <u>are</u> numbers of arithmetic, the '$|\ |$'s occurring in the displayed sentences must refer to the operation A--for, for any points O, L, and U, the measures of $\overleftrightarrow{OL}$ and $\overleftrightarrow{OU}$ are numbers of arithmetic. So, here, and in the pages which follow, the context implicitly prohibits the interpretation of '$|\ |$' as referring to B.

In summary:

1. (a) It is customary to use those numerals which may name numbers of arithmetic in two ways--sometimes as names for numbers of arithmetic, and sometimes as names for the corresponding non-negative real numbers.

   (b) Numerals in which the principal operator is '$|\ |$' are of this kind. So, we use '$|\ |$' sometimes for 'A(. . .)', and sometimes for 'B(. . .)'.

2. (a) We adopt the convention that, after page 1-32 such "ambiguous numerals" will, in fact, be interpreted as naming the corresponding nonnegative real numbers whenever the context does not, explicitly or implicitly, prohibit this interpretation.

   (b) So, in Unit 2 and later units, '| |' refers to the operation B unless the context prohibits this interpretation. In Unit 1, from page 1-104 through the first half of page 1-110, the text specifies that each occurrence of '| |' refers to the operation A. The last half of page 1-110 calls attention to the possibility of interpreting '| |' as a name for B. It would probably be a help to students if the discussion on page 1-110 were extended as follows:

   > From now on, in most of your work with absolute values you will want to use the real number interpretation. So, let's agree that:
   >
   > > Numerals which contain a '| |' should be interpreted as numerals for real numbers--except in places where the context prohibits this interpretation.

3. Since the same symbol '| |' is used to name both operation A and operation B, it is natural to use the same words, 'absolute valuing' to name both operations, and, in all cases, to read '|...|' as 'the absolute value of...'. So the meaning of each occurrence of the words 'absolute value' must, like the interpretation of '| |' be determined from the context in which it occurs. As a matter of fact, this has been the case traditionally. Texts used in high schools often give rules similar to the following:

   > To add two numbers with the same sign, add their absolute values and prefix the common sign.

   Here, the words 'absolute value' refer to operation A. But, when a student gets to advanced work he is given the following definition:

   $$\forall_{x \geq 0}\ |x| = x \qquad \text{and} \qquad \forall_{x < 0}\ |x| = -x,$$

   and is told to read '| |' as 'the absolute value of'. So, here, he is to use 'absolute value' to refer to operation B.--A. H.

A VERY SHORT SHORT-COURSE

Many teachers of UICSM materials are receiving requests to conduct inservice training courses for teachers. In some cases the time allotted for this training is extremely short. A request for advice as to what to do in such a situation was recently received here. Our response to this request is embodied in the paragraphs which follow.

You have been asked to conduct a training course for those persons who are teaching or will teach the first four units in your school!

You are attempting an impossible task if you try to really teach the four units in the time allotted. You may cover the pages but you won't uncover much. I would select several topics and ignore the rest except for questions that are raised. Since so very many of the principles which are used in the last three units are foreshadowed in the first unit, I suggest that you take particular care with Unit 1. Remember that the fill-in exercises are very important in clarifying the principles.

You will notice that the sample outline which follows suggests that at least three sessions be devoted to Unit 1 alone. Since your course is of such short duration, you should insist that each person who will take the course have a copy of Unit 1 before the first session. If the participants read the Introduction of Unit 1 prior to the first session, you will be able to get into the "heart" of the unit sooner.

Session I

1. Read the Introduction [if necessary] and discuss the exercises.
2. Do not use time here for a long discussion of the value of the Introduction. Bring this into later discussions of topics where misunderstanding of written material could occur if we did not have a way of distinguishing between a symbol and its referent.
3. Emphasize that no more than two days should be spent on the Introduction with a class of 8th or 9th grade students.
4. Assign the first half of Unit 1 for the next session. Be sure to tell your group that they should not expect to grasp all the material in the TC in one reading.

Session II

1. Develop the conventions we use for abbreviating and unabbreviating expressions.
2. Discuss the idea of the principal operator in an expression.
3. Develop pattern sentences for the basic principles. [The first Newsletter contains material that should help on principal operator and pattern-sentence.]

4. Be sure the teachers understand that

   '(2 + 7) + 3' and '2 + 7 + 3'

   name the same number because of an <u>abbreviating</u> convention. On the other hand, the expressions

   '(2 + 7) + 3' [or, '2 + 7 + 3']

   and '2 + (7 + 3)'

   name the same number by virtue of the associative principle for addition.

5. Point out that

   '(8 ÷ 4) ÷ 2' and '8 ÷ 4 ÷ 2'

   name the same number by convention, but that

   '(8 ÷ 4) ÷ 2' [or, '8 ÷ 4 ÷ 2']

   and '8 ÷ (4 ÷ 2)'

   do NOT name the same number. [There is no associative principle for division.]

6. Assign the last half of Unit 1 for the next session.

<u>Session III</u>

1. Distinguish between sentences which are instances of the principles, and those which are consequences but <u>not</u> instances. [Of course, any instance is a consequence.]

2. Study derivations of such sentences as:

   $$98 \times 20 + 9 \times 98 + 98 \times 21 = (21 + 9 + 20) \times 98$$

3. It is wise to do a careful job with derivations of the kind mentioned in Item 2 above, since this will make the work in Unit 2 much easier.

4. Assign the first half of Unit 2 for the fourth session.

<u>Session IV</u>

1. Discuss operations and their inverses. [Point out that certain operations do not have inverses.] Be sure to include the operations opposition, nonpositiving, nonnegativing, and both absolute valuing operations in this discussion. [See the first article in this newsletter for a discussion of the two operations each of which is named 'absolute valuing'.]

2. Stress the manner in which open <u>sentences</u> can be used to generate other sentences, and the way pronumeral <u>expressions</u> enable one to generate other expressions.

3. Page 2-21 usually requires careful attention. The use of the principal operator is helpful here.

4. Discuss the pattern for testing statements of the forms:

$$(3x + 7y) + (5x + 6y) = (3 + 5)x + (7 + 6)y$$

and:

$$(3x)(7x) = (3 \cdot 7)(xx)$$

Make certain that teachers understand the role the test-pattern plays in a proof.

5. Assign theorems to be proved.

Session V

1. By this time you should be far enough behind that you will need this session to catch up with the material outlined for the first four sessions. You may want to use the next session this way also.

2. Assign the last half of Unit 2.

Session VI

1. Work on proofs.

2. Assign more theorems to be proved. Have both Unit 2 and Unit 3 brought to class for Session VII.

Session VII

1. Develop the intuitive approach to solving equations and inequations.

2. Introduce the concepts of solution sets, loci, and the use of brace notation for naming sets.

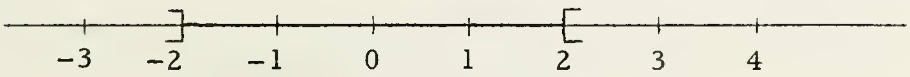
3. Emphasize that at this time the student should have no formal rules for solving equations and inequations.

4. Get the teachers involved in solving:

$$|x| = 2$$
$$|x| < 2$$
$$|x| \geq 2$$

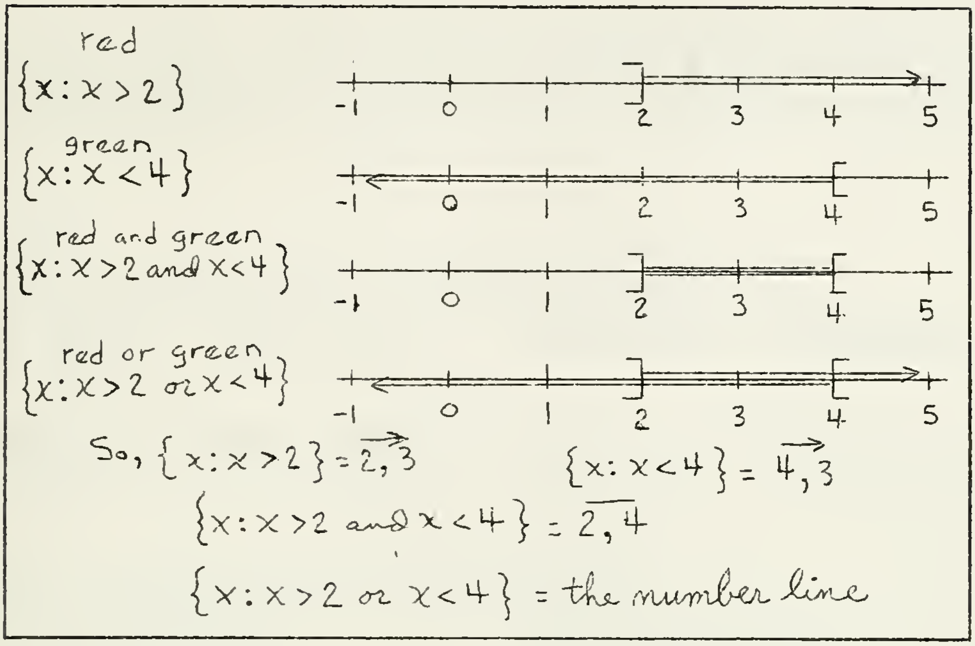
intuitively. See that many different names are given for each solution set. Include the graphing of the solution set. Thus:

$$\{x: |x| < 2\} = \overline{{}^-2, 2} = \{x: {}^-2 < x < 2\} = \{x: x^2 < 4\}$$



We hope to have an article in the next newsletter which will help here.

5. Be certain that the use of 'and' and 'or' is clear. Use colored chalk. Your blackboard may look like this:



6. **Assign the first half of Unit 3.**

<u>Session VIII</u>

1. Devote this session to the transformation principles for equations. A comparison of "transposition" and addition transformation principle may be profitable.
2. Stress the importance of the factoring transformation principle and its use in solving quadratics by completing the square.
3. Assign the last half of Unit 3.

<u>Session IX</u>

1. Answer any questions about the last half of Unit 3.
2. Start the discussion of ordered pairs and graphing. Graph:

   $\{(x, y): x > 2\}$; $\{(x, y): y < 3\}$;

   $\{(x, y): x > 2 \text{ and } y < 3\}$; $\{(x, y): x > 2 \text{ or } y < 3\}$.
3. Assign Unit 4.

<u>Session X</u>

1. Discuss any material about which questions are raised.
2. Omit Unit 4 <u>completely</u> if necessary.

Good luck with your in-service courses. We hope this outline will be helpful.--A. H.

UICSM-NETRC Math Study Tests

In Newsletter No. 1 we presented the first four tests in the UICSM-NETRC series. Here are the next four, designated by the letters E, F, G, and H. The page to have been completed is again in brackets after the letter. A fifteen minute time limit has been set for each for the purposes of the NETRC Math Study. --R.S.

Test E [1-72]

1. The operation multiplying by 2 does <u>not</u> contain:
   (A) (1, 2) (B) (0, 1) (C) (2, 4) (D) ($\frac{1}{2}$, 1)

2. The inverse of adding 5 contains:
   (A) (1, 5) (B) (5, 5) (C) (5, 0) (D) (0, 5)

3. Another name for the inverse of adding 0 is:
   (A) multiplying by 0 (B) subtracting 1
   (C) multiplying by 1 (D) none of these

4. Which pair belongs both to adding 3 and to adding 4?
   (A) (3, 4) (B) (1, 8) (C) (4, 7) (D) none of these

5. Which pair belongs both to multiplying by 2 and to adding 5?
   (A) (5, 10) (B) (1, 7) (C) (10, 5) (D) none of these

6. Which pair belongs both to multiplying by 3 and to the inverse of multiplying by 3?
   (A) (0, 3) (B) (0, 0) (C) (1, 1) (D) none of these

7. Which pair belongs both to adding 4 and to the inverse of adding 4?
   (A) (0, 4) (B) (4, 0) (C) (0, 0) (D) none of these

8. Which pair belongs both to the inverse of multiplying by 5 and to the inverse of adding 12?
   (A) (15, 3) (B) (3, 15) (C) (60, 0) (D) none of these

9. The pair (18, 25) belongs to an adding operation. Which pair belongs to its inverse?
   (A) (16, 9) (B) (8, 15) (C) (0, 0) (D) none of these

10. Which of these operations is not the same as dividing by 5 ?

(A) the inverse of multiplying by 5

(B) the inverse of multiplying by the reciprocal of 5

(C) multiplying by the reciprocal of 5

(D) the inverse of dividing by the reciprocal of 5

11. Which of these is a true principle for numbers of arithmetic ?

(A) commutative principle for division

(B) associative principle for division

(C) distributive principle for division over division

(D) none of them

12. Which of these is not a true principle for numbers of arithmetic ?

(A) commutative principle for addition

(B) associative principle for addition

(C) distributive principle for addition over addition

(D) principle for adding zero

*

Choose the principle for real numbers which justifies each sentence.

13. $^{-}18 \times 0 \times 47 + 1 = 0 \times 47 + 1$

(A) pm1 (B) pm0 (C) pa0 (D) apm

14. $^{-}18 \times 47 \times 0 + 1 = {}^{-}18 \times (47 \times 0) + 1$

(A) apm (B) pm0 (C) pm1 (D) pa0

15. $(1 + {}^{-}7) \times {}^{-}8 + 2 \times {}^{-}8 = (1 + {}^{-}7 + 2) \times {}^{-}8$

(A) cpa (B) apa (C) dpma (D) apm

16. $({}^{-}7 + {}^{-}3) \times ({}^{-}9 + 10 + {}^{-}2 + 6) = ({}^{-}7 + {}^{-}3) \times [({}^{-}9 + 10) + ({}^{-}2 + 6)]$

(A) apa (B) cpa (C) cpm (D) apm

17. $^{-}1 \times 1 \times 0 + 0 = 1 \times {}^{-}1 \times 0 + 0$

(A) pm0 (B) pm1 (C) cpm (D) pa0

18. $1 \times (0 + 1) \times 0 = (1 \times 0 + 1 \times 1) \times 0$

(A) pm1 (B) ℓdpma (C) pm0 (D) pa0

19. $(0 + 0 \times 1) \times 1 + 0 = (0 + 0) \times 1 + 0$

(A) apm (B) pa0 (C) pm0 (D) pm1

20. $(0 \times 1 + 0) \times 1 + 0 = 0 \times 1 \times 1 + 0$

(A) apm (B) pm1 (C) pm0 (D) pa0

Key for Test E [1-72]

1. B	2. C	3. C	4. D	5. A	6. B	7. D
8. A	9. A	10. B	11. D	12. C	13. B	14. A
15. C	16. A	17. C	18. B	19. D	20. D	

Test F [1-92]

I. Choose the answer which will make the sentence true.

1. $-(3 - 8) = $ ______

(A) 5 (B) −5 (C) 11 (D) −11

2. $- - -3 = $ ______

(A) −9 (B) 9 (C) −3 (D) 3

3. $-13 - -7 = $ ______

(A) −6 (B) 6 (C) −20 (D) 20

4. $4 - 9 - 7 + 13 = $ ______

(A) 1 (B) −1 (C) −7 (D) none of these

5. $5 - -8 - 2 + -3 - 1 = $ ______

(A) −9 (B) 7 (C) 13 (D) none of these

6. $-(8 + $ ______ $) = 10$

(A) −18 (B) 18 (C) −2 (D) 2

7. $-(7 - $ ______ $) = 3$

(A) −10 (B) 10 (C) −4 (D) 4

8. $-(5 + $ ______ $) - 8 = 0$

(A) 3 (B) −3 (C) 13 (D) none of these

9. $(2 - $ ______ $) \times -5 = -40$

(A) 10 (B) −10 (C) −6 (D) none of these

10. $3 - \underline{\qquad} \times (4 - 10) = 15$

(A) $\frac{11}{2}$ (B) -18 (C) 2 (D) none of these

II. Choose the correct answer.

11. Which number is the opposite of itself?

(A) 1 (B) $^{-}1$ (C) 0 (D) none of these

12. Which kind of problem does not always have a solution?

(A) subtracting numbers of arithmetic (B) subtracting real numbers
(C) adding numbers of arithmetic (D) adding real numbers

13. The inverse of adding the opposite of 5 is the same as

(A) adding 5 (B) subtracting 5 (C) adding $^{-}5$ (D) none of these

14. Subtracting the opposite of $^{-}8$ is <u>not</u> the same as

(A) adding the opposite of 8 (B) adding $^{-}8$
(C) subtracting 8 (D) subtracting $^{-}8$

15. If a first real number is the opposite of a second real number, then

(A) their difference is nonnegative (B) their product is nonpositive
(C) their quotient is 1 (D) none of these

16. Pick a real number, then add it to its opposite, then multiply the sum by the number picked. What kind of real number will the product be?

(A) positive (B) negative (C) 0 (D) cannot tell

17. Which operation is the inverse of oppositing?

(A) oppositing (B) adding 0 (C) multiplying by 0 (D) none of these

18. A first number is not necessarily the opposite of a second when

(A) the opposite of the first number is the opposite of the second
(B) the sum of the numbers is 0
(C) the first number is the product of the second number by $^{-}1$
(D) the second number is the product of the first number by $^{-}1$

19. The opposite of the sum of a first number and a second number is

    (A) the first number minus the second number

    (B) the second number minus the first number

    (C) the sum of their opposites

    (D) the opposite of the first number, plus the second number

20. After a first number and a second number have been chosen, which of these is not always the same as the others?

    (A) the first number minus the opposite of the opposite of the second number

    (B) the first number plus the opposite of the second number

    (C) the first number minus the second number

    (D) the first number plus the second number

Key for Test F [1-92]

1. A	2. C	3. A	4. A	5. B	6. A	7. B
8. D	9. C	10. C	11. C	12. A	13. A	14. D
15. B	16. C	17. A	18. A	19. C	20. D	

Test G [2-22]

I. No more than one answer to each problem is a true sentence. Which is it, if any?

1. (A) $|3 - 10| = 3 - 10$ (B) $|3 - 10| = {}^{-}|10 - 3|$
   (C) $|3 - 10| = |10 - 3|$ (D) none of these

2. (A) $3 + {}^{-}3 < 3 - {}^{-}3$ (B) $3 + {}^{-}3 \neq -(3 + {}^{-}3)$
   (C) $2 - 3 > 3 - 2$ (D) none of these

3. (A) $0 \leq 0$ (B) $3 - {}^{-}10 = 3 + -10$
   (C) $(2 - 3) + -(3 - 2) = 0$ (D) none of these

4. (A) '$3x - 3$' is a numeral (B) $x + y = y + x$
   (C) '$x \cdot 0 = 0$' is an open sentence (D) none of these

5. (A) $a + a = 2a$ (B) $b + 0 = b$
   (C) $c \cdot 1 = c$ (D) none of these

II. Which substitution converts the open sentence into a true one?

6. $3x = x + 6$

(A) ‘3’ for ‘x’ (B) ‘0’ for ‘x’ (C) ‘y - 6’ for ‘x’ (D) none of these

7. $d \cdot 0 = 0$

(A) any numeral for ‘d’ (B) ‘x’ for ‘d’

(C) ‘x + −x’ for ‘d’ (D) none of these

8. $x + y = 0$

(A) ‘−x’ for ‘y’ (B) ‘3’ for ‘x’ and ‘−3’ for ‘y’

(C) ‘−y’ for ‘x’ (D) none of these

9. $2x - 5y = 34$

(A) ‘2’ for ‘x’ and ‘−6’ for ‘y’ (B) ‘22’ for ‘x’ and ‘−2’ for ‘y’

(C) ‘−2’ for ‘x’ and ‘6’ for ‘y’ (D) none of these

10. $xy = yx$

(A) any numeral for ‘x’ (B) ‘x’ for ‘y’

(C) any numeral for ‘y’ (D) none of these

III. Which substitution converts the open sentence into a <u>false</u> one?

11. $xx - 5x + 4 = 0$

(A) ‘4’ for ‘x’ (B) ‘1’ for ‘x’

(C) ‘0’ for ‘x’ (D) none of these

12. $(x - 2)(x + 3)(x - 5) = 0$

(A) ‘2’ for ‘x’ (B) ‘−3’ for ‘x’

(C) ‘5’ for ‘x’ (D) none of these

IV. From which pattern-expression can the given expression be obtained by substitution?

13. $6 + 3x$

(A) $6 + xx$ (B) $x + y$ (C) $2v + vx$ (D) $(u + v)x$

14. $3x + 5x$

(A) $ax + bx$ (B) $xx + yx$ (C) $(y + z)x$ (D) $3x + xy$

15. $3a(b - 2c)$

(A) $xa(y - xc)$ (B) $3x(x - y)$ (C) $xy - z$ (D) xy

16. $3(x - 1)(x - 4)$

(A) $x(x - 1)(x - 4)$ (B) xyz

(C) $b(x - a)(x - b)$ (D) $x(yz)$

17. $9 + 3(x - y)$

(A) $x + 3y$ (B) $x + xy$

(C) $x + y(x - y)$ (D) $z + x - y$

18. $3a + 6b + c$

(A) $uv + wx + y$ (B) $x + (y + z)$

(C) $(3x + 6)(y + z)$ (D) $xy + 2xz + c$

19. $(2a + 10b)(cb)$

(A) $(u + v)wx$ (B) $(x + y)z$

(C) $(xa + 5xy)z$ (D) $x + yz$

20. $x(x + y) + y(x + y)$

(A) $yx + zx$ (B) $xy + yz$

(C) $xy + yy$ (D) $u(v + y) + v(v + y)$

Key for Test G [2-22]

1. C	2. A	3. A	4. C	5. D	6. A	7. A
8. B	9. A	10. D	11. C	12. D	13. B	14. A
15. D	16. B	17. A	18. A	19. B	20. A	

Test H [2-38]

I. Of which principle is the given generalization a consequence?

1. For each a, $3 + a = a + 3$.

(A) apa (B) cpa (C) cpm (D) none of these

2. For each b, $(3 - b) + -(3 - b) = 0$

(A) ps (B) cpa (C) apa (D) none of these

3. For each c, $c = c \cdot 1$.

(A) cpm (B) apm (C) pm1 (D) none of these

4. For each d, $d \cdot 1 + d \cdot 1 = d(1 + 1)$

(A) ℓdpma (B) dpma (C) apm (D) none of these

5. For each e, $10e - 3e = 10e + -(3e)$.

(A) po (B) ps (C) apa (D) none of these

6. For each f, $3f + (7f + f) = 3f + 7f + f$.

(A) ℓdpma (B) cpa (C) apa (D) none of these

7. For each g, $(g - 3)g = (g - 3)(g + 0)$.

(A) ps (B) dpma (C) pm0 (D) none of these

8. For each h, $3h + (3h + 4h)h = 3h + (4h + 3h)h$.

(A) dpma (B) apa (C) cpa (D) none of these

9. For each u, $u(u \cdot 1) + 0 = uu \cdot 1 + 0$.

(A) apm (B) pm1 (C) pa0 (D) none of these

10. For each x, for each y, $(x + y)y \cdot 0 = (xy + yy) \cdot 0$.

(A) dpma (B) apm (C) pm0 (D) none of these

II. Which answer will make the completed generalization true?

11. For each a, $2a + 3a =$ ______.

(A) 5aa (B) 6aa (C) $5(a + a)$ (D) none of these

12. For each x, $x(x + 3) =$ ______.

(A) $xx + 3$ (B) $2x + 3$ (C) $xx + 3x$ (D) none of these

13. For each m, $(3m - 10) - (3x - 10) =$ ______.

(A) 0 (B) $6m - 20$ (C) $-6m + 10$ (D) none of these

14. For each x, $\frac{1}{3}x + \frac{1}{2}x =$ ______.

(A) $\frac{5}{6}x$ (B) $\frac{2}{5}x$ (C) $\frac{7}{12}x$ (D) none of these

15. For each w, $(w + 2) + (w + 3) + (w + 4) =$ ______.

(A) $www + 9$ (B) $w + 9$ (C) $3w + 9$ (D) none of these

16. For each k, $(2k + 3)(k + 1) =$ ______.

(A) $5k + 1$ (B) $3k + 4$ (C) $2kk + 3$ (D) none of these

17. For each t, $(2t)(3t)(5t) =$ _____.

(A) 10t (B) 10ttt (C) 30t (D) none of these

18. For each u, $2(3u + 1)[5(4 + u)] =$ _____.

(A) $7(3u + 1)(4 + u)$ (B) $10(4u + 5)$

(C) $10(u + 4)(3u + 1)$ (D) none of these

19. For each v, $3v + (v + 2)v =$ _____.

(A) 6v (B) vv + 5v (C) $[3v + (v + 2)]v$ (D) none of these

20. For each y, $y(y + 1) + y(y + 2) =$ _____.

(A) $y(2y + 3)$ (B) yy + 3y (C) 2yy + 3 (D) none of these

Key for Test H [2-38]:

1. B	2. D	3. C	4. A	5. B	6. C	7. D
8. C	9. A	10. A	11. D	12. C	13. A	14. A
15. C	16. D	17. D	18. C	19. B	20. A	

* * *

Recommended Reading

Two books which we think would be of special interest and value to UICSM teachers are:

Max Beberman <u>An Emerging Program of Secondary School Mathematics</u> 1958

Jerome S. Bruner <u>The Process of Education</u> 1960

Both were published by the Harvard University Press.

Two paperbacks from which all mathematics teachers should profit are:

Irving Adler <u>The New Mathematics</u> Mentor Books, 1959 (50¢)

G. Polya <u>How to Solve It</u>: A New Aspect of Mathematical Method Second Edition. Doubleday Anchor Books, 1957 (95¢)

* * *

Quiz - Unit 4, page 4-41

A. Suppose $A = \{-3, -2, -1, 0, 1, 2, 3\}$
$B = \{-5, -4, -3, -2, -1, 0, 1, 2, 3\}$

1. How many ordered pairs are there in $A \times B$? ______

2. How many ordered pairs in $A \times B$ have first component -2? ______

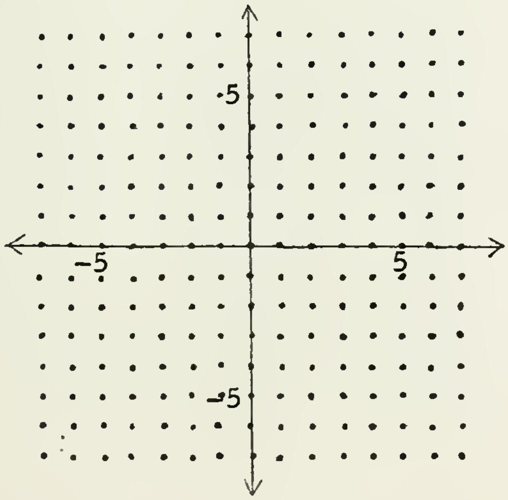
3. How many ordered pairs in $A \times B$ have second component greater than or equal to 0? ______

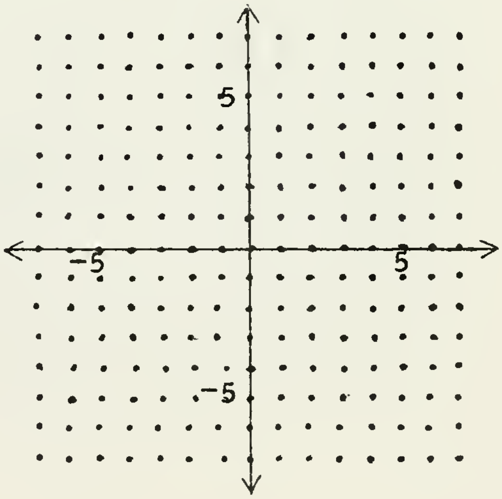
4. How many ordered pairs in $A \times B$ have first component less than -1 <u>and</u> second component greater than 0? ______

5. How many ordered pairs in $A \times B$ have first component less than -1 <u>or</u> second component greater than 0? ______

B. On a picture of the number plane lattice indicate the dots which are pictures of ordered pairs of integers such that the first component is 1 more than the second component.

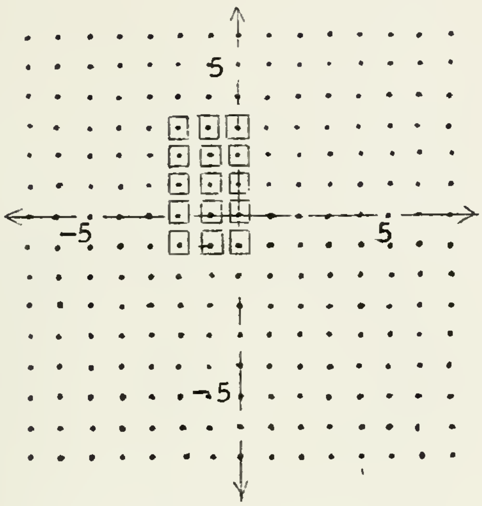
C. On a picture of the number plane lattice indicate the dots which are pictures of ordered pairs of integers such that the sum of the components of each ordered pair is 3.





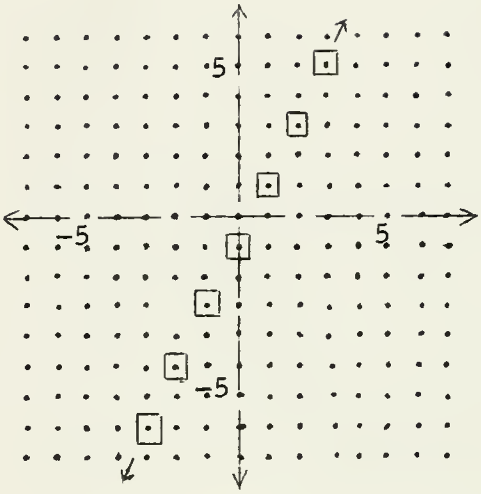
D. Use brace-notation to describe the sets pictured below

(1)



{________________}

(2)



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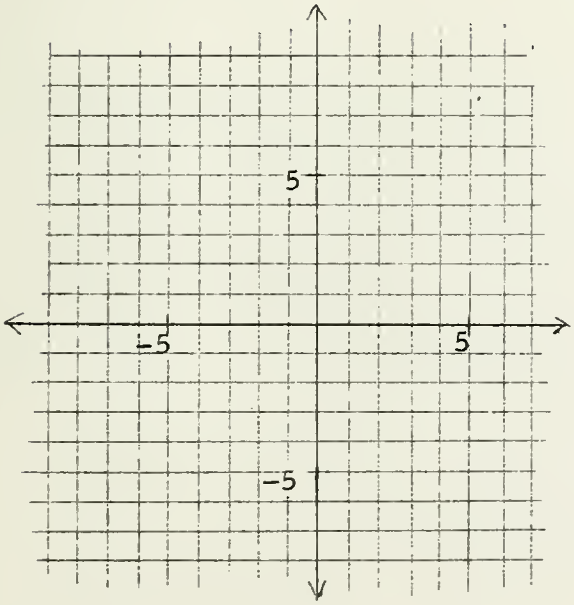
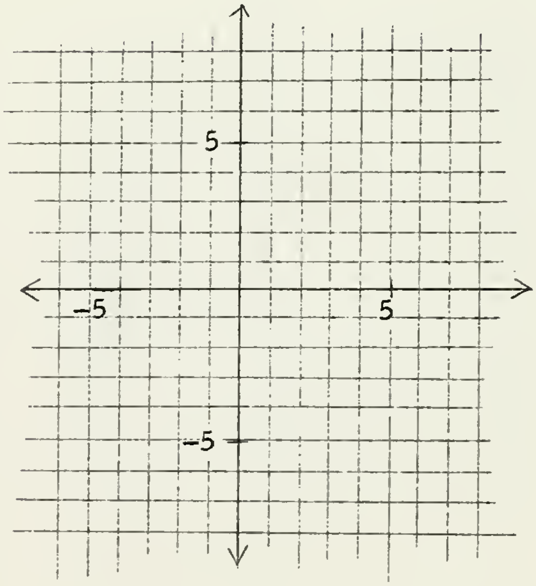
E. On a picture of the number plane graph the following:

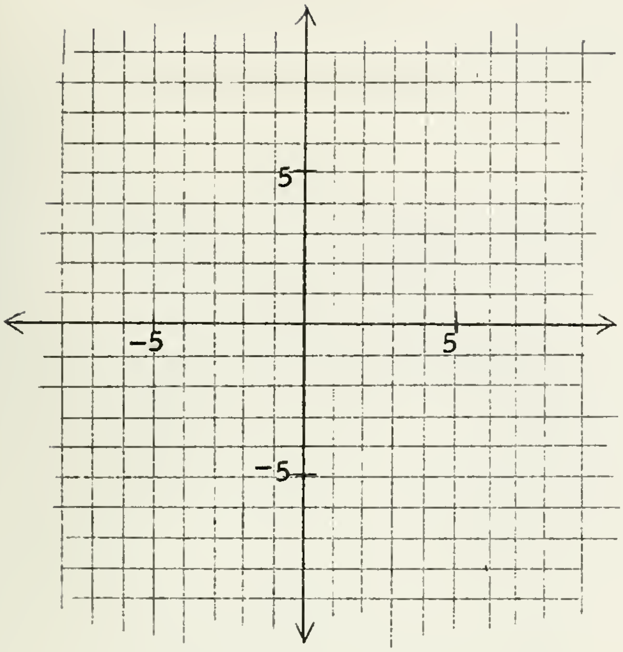
(1) $y = 3$



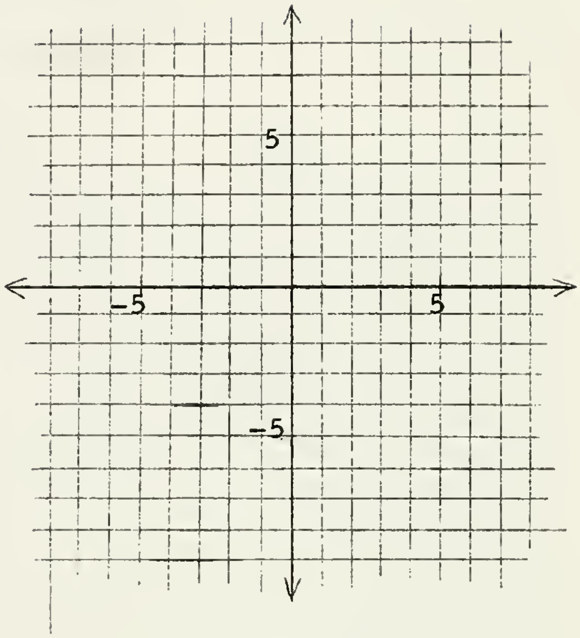
(2) $x < 2$



(3) $y = x + 2$



(4) $|x| < 5$



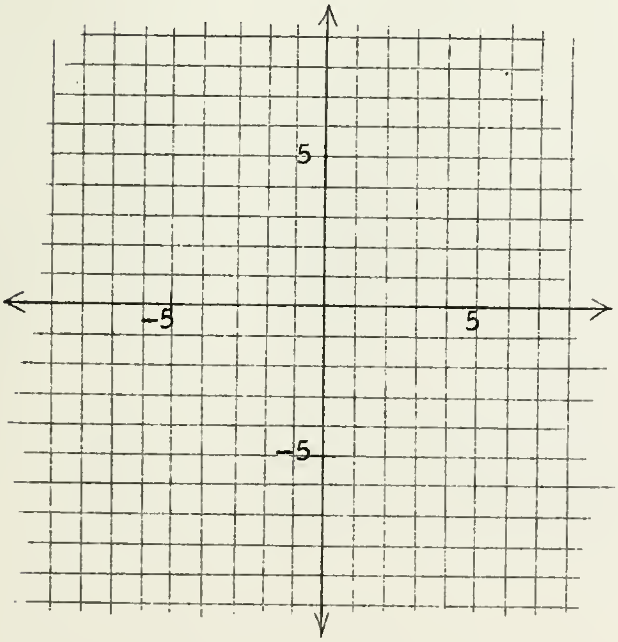
F. Graph each of the two equations and give the ordered pairs which are in the intersection of their solution sets.

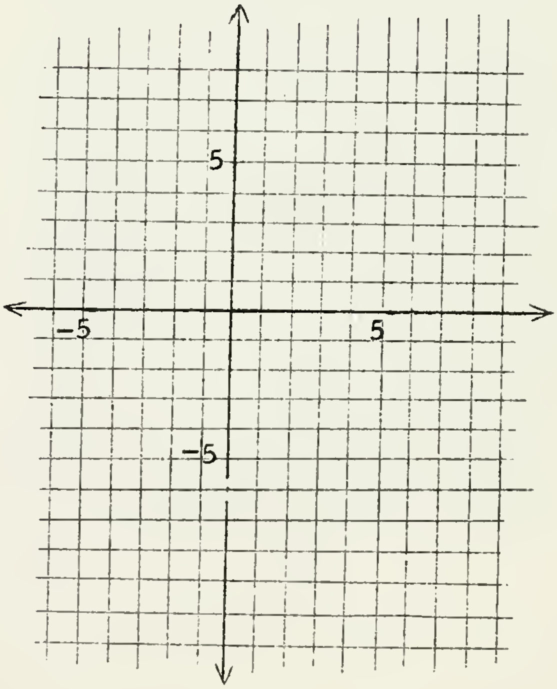
1. (a) $x + y = 3$

   (b) $2x + y = 6$

2. (a) $|x - 2| = y$

   (b) $|y| = 2$





G. Graph:

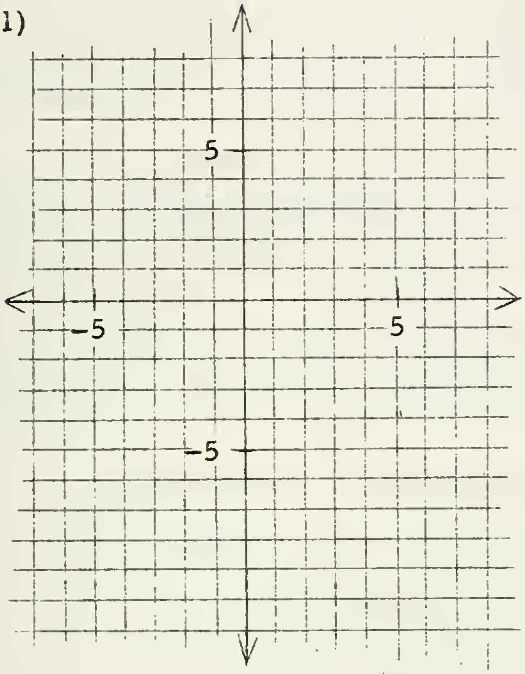
1. $\{x, y): x \leq 2\}$

2. $\{(x, y): |y| > 3\}$

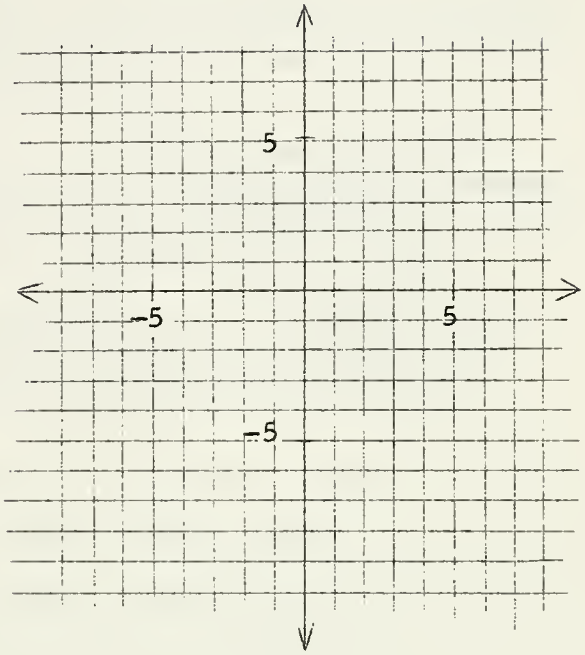
3. $\{(x, y): x \leq 2 \text{ and } |y| > 3\}$

4. $\{(x, y): x \leq 2 \text{ or } |y| > 3\}$

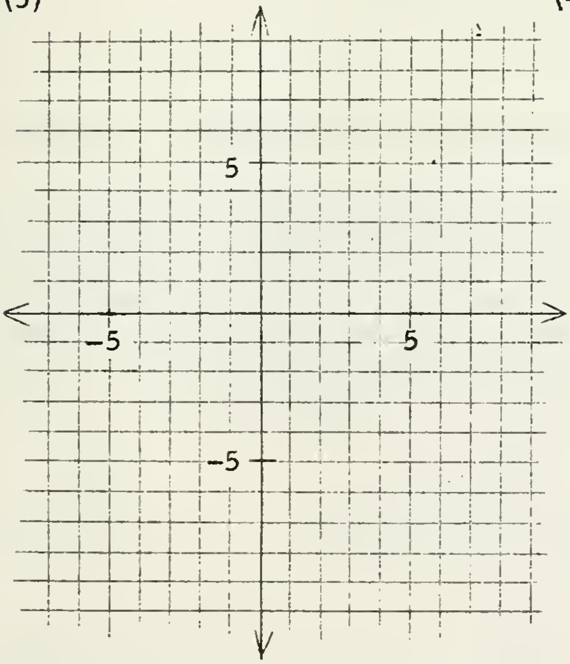
(1)



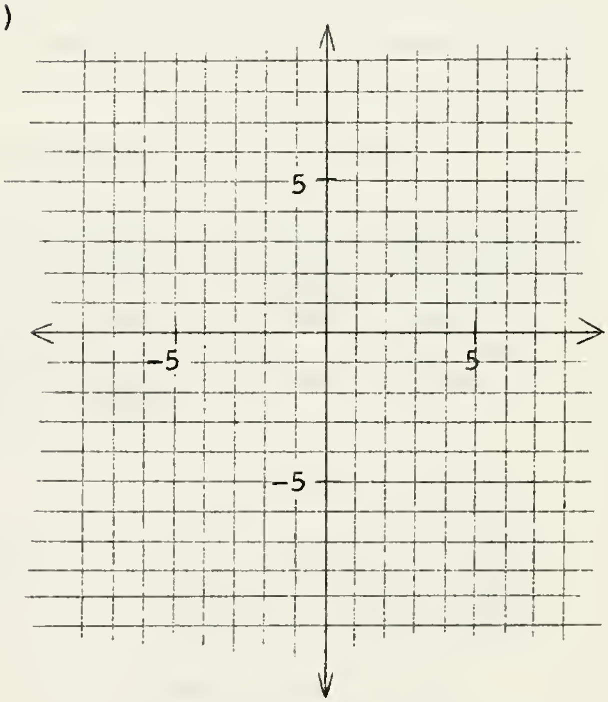
(2)



(3)



(4)



NEWS AND NOTICES

Mr. William Annett of Seaford High School, Seaford, L. I., New York, a new UICSM teacher and a member of the NETRC Film Study experimental group, will speak on First Course at the annual convention of the New York State Mathematics Teachers Association.

Mr. Donald E. Palzere of the Edwin O. Smith School at the University of Connecticut, Storrs, was a panelist at the State Teacher's Convention in New Haven on October 28. He spoke as the UICSM representative in a presentation of new programs in school mathematics.

Mrs. Mary Huzzard of Jenkintown, Pennsylvania, participated in a panel at Villanova University on October 28 in connection with Mr. Beberman's appearance there. Mrs. Huzzard teaches at both Cheltenham Senior High School, Wyncote, and Ogontz Junior High School, Elkins Park.

Mr. Beberman will speak at a meeting of the Illinois Association of School Administrators in Chicago on November 21. He subsequently will visit the following schools in addition to those listed in the last Newsletter:

November 30 -- Permian High School, Odessa, Texas

December 1 & 2 -- Tucson, Arizona, public schools

5 -- San Diego, California, public schools

6 -- Desert Sun School, Idyllwild, California

7 -- E. M. Cope Junior High School, Redlands, Calif.

REGIONAL ORIENTATION CONFERENCES IN MATHEMATICS

A series of regional conferences designed to acquaint administrators and supervisors with curriculum revision projects such as UICSM has been taking place across the nation this autumn under the sponsorship of the National Council. The format of each meeting includes a panel of teachers who comment on their experiences in teaching one of the new courses. ROCM Regional Directors have supplied the names of the teacher-panelists representing UICSM at several of these conferences.

Dates	Location	UICSM Representative
October 3-4	Philadelphia, Pa.	Sister Mary of the Angels St. Rosalia High School Pittsburgh, Pennsylvania
October 10-11	Iowa City, Iowa	Miss Grace Wandke Barrington High School Barrington, Illinois

Dates	Location	UICSM Representative
October 27-28	Atlanta, Georgia	Miss Emma Mae Large Mary Holmes Junior College West Point, Mississippi
November 3-4	Portland, Oregon	Miss Dana Small Franklin High School Portland, Oregon
November 18-19	Los Angeles, Calif.	Stewart Moredock Sacramento State College Sacramento, California
December 9-10	Miami, Florida	Robert Kansky Melbourne High School Melbourne, Florida
December 15-16	Cincinnati, Ohio	Eugene Epperson Talawanda High School Oxford, Ohio

Some Statistics

Miss Eleanor McCoy, Associate Teacher Coordinator for the Project, reports the following data on cooperating schools for the 1960-61 school year: there are 144 such schools, located in 94 cities and in 28 states. 274 teachers teach 521 UICSM classes with a total enrollment of 12,186 students. A breakdown by year of study is given below.

Year of study in UICSM mathematics	No. of classes	Enrollment
1st	299	7109
2nd	142	3510
3rd	66	1324
4th	12	226
5th	2	17

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'SIMPLIFY'

The word 'simplify' is used in so many ways that it is impossible to decide that any particular expression is simpler than another equivalent expression. Whenever you read 'simpler' ask: "Simpler for what purpose?"

(1) How many pennies are in the box? If the number of pennies is fifty-seven, we would all feel that '57' is a better name than '$3 \cdot 19$'.

But consider the question: If nineteen pencils cost fifty-seven cents, how much does each pencil cost? If we use the name '$3 \cdot 19$' in our thoughts it is much easier to realize immediately that each pencil costs 3 cents. This is a case in which '$3 \cdot 19$' is a simpler name for 57 than '57' is.

(2) Which of these two equivalent expressions is the simpler?

$$2(19x - 7)$$

$$38x - 14$$

If we are concerned with the number of operations involved after a numeral replaces 'x', we see that the expression:

$$2 \cdot (19 \cdot 3 - 7)$$

involves $19 \cdot 3$, $57 - 7$, and $2 \cdot 50$; we have three operations here. On the other hand, the expression:

$$38 \cdot 3 - 14$$

involves $38 \cdot 3$ and $114 - 14$--only two operations.

If we replace 'x' by '3' and evaluate expressions such as:

(1) $$\frac{38x - 14}{19x - 7}$$

and:

(2) $$\frac{2(19x - 7)}{19x - 7}$$

we find that (1) involves five operations while (2) can be simplified quickly by using:

$$\forall_x \forall_{y \neq 0} \qquad \frac{xy}{y} = x$$

So, for this purpose, we would consider '$2(19x - 7)$' simpler than '$38x - 14$'.

We should begin (as soon as possible) to teach the child that the meaning of the word 'simplify' must be considered relative to the problem in which

we are engaged. Try to do this by stressing the principal operation that is involved in an expression. The first time (and many other times, for a period of several days or even weeks) that we consider the theorem:

$$\forall_x \forall_{y \neq 0} \forall_{z \neq 0} \frac{x \cdot z}{y \cdot z} = \frac{x}{y}$$

point out that in order to use this generalization we must have multiplication as the principal operation for both numerator and denominator.

Long before this, a discussion similar to the following should have occurred.

Consider the pattern sentence:

(1) $$\underline{a \cdot (b + c)} = \underline{\underline{a \cdot b + a \cdot c}}$$

What is the principal operation indicated in the expression which is underlined once? What is the principal operation in the expression which is underlined twice? So (1) gives us a pattern which we can use to find an expression in which the principal operation is addition and which is equivalent to a given expression in which the principal operation in multiplication. Sentence (1) also gives us a pattern:

$$a \cdot b + a \cdot c = a \cdot (b + c)$$

which we can use to find an expression in which the principal operation is multiplication and which is equivalent to a given expression in which the principal operation is addition.

In some cases it is more convenient that the principal operation be multiplication. In others, we can do our work more rapidly if the principal operation is addition. Much of the practice that we are doing now will prepare us for more complicated problems later. When that time comes, we must decide whether it will be <u>simpler</u> to use an expression such as:

$$a \cdot (b + c)$$

or to use an expression such as:

$$ab + ac$$

We can only decide this after we have encountered the problem. At that time we must be able to change from one expression to an equivalent one rapidly.

To go back to:

$$\forall_x \forall_{y \neq 0} \forall_{z \neq 0} \frac{x \cdot z}{y \cdot z} = \frac{x}{y}$$

If we try to 'reduce a fraction' we must first be certain that the principal operation in both numerator and denominator is multiplication. You may find that this is a very effective way to help students avoid the following error:

$$\frac{ac + bc}{c} = ac + b$$

Of course, this work on principal operation starts back in Unit 1 when we are finding unstructured names for such numbers as $3 + 8 \cdot 7$ and $(3 + 8) \cdot 7$.

When we get to an exercise such as:

Simplify: $\frac{3}{7x} + \frac{2}{5y}$

we point out that for some purposes we consider an expression in which the principal operation is division simpler than one in which the principal operation is addition. So, such exercises are designed to give us practice in making such changes quickly and correctly.

Some of the difficulty arising from the use of the word 'simplify' is due to the child's earlier training. We are all inclined to feel that we have not multiplied 3 by 19 unless we write (or say) '57'. When we see '3·19', we construe this as a command to find the equivalent standard decimal numeral (which is '57'). As arithmetic teachers we say

What is 3 times 19?

and we expect '57' for the answer to the question. We mean:

What is the standard decimal numeral for the product of 3 by 19?

We must begin our study of the word 'simplify' as soon as possible. If we can begin in elementary school, so much the better. If we have to start in the higher grades, let's start then.--A.H.

ANOTHER USE FOR PRINCIPAL OPERATOR

The use of the principal operator (Newsletter 1) to determine if a certain sentence is an instance of a given principle is actually a rather complicated form of another use of the principal operator. To illustrate:

In order to show that:

(*) $$7 + (3 + 2) = (7 + 3) + 2$$

is an instance of the associative principle for addition, we examine the pattern sentence:

(**) $$(\underline{\qquad} + \text{- - -}) + \backsim\backsim\backsim = \underline{\qquad} + (\text{- - -} + \backsim\backsim\backsim).$$

We must decide whether or not either side of (*) will "fit" the first side of (**). Having decided that the second side of (*) "fits" the first side of (**), we must then decide whether or not (with the same substitutions) the first side of (*) "fits" the second side of (**). In effect, our first step is to examine the two <u>pattern</u> <u>expressions</u>.

(a) $(\underline{\qquad} + \text{- - -}) + \backsim\backsim\backsim$ (b) $\underline{\qquad} + (\text{- - -} + \backsim\backsim\backsim)$

and then to determine if either '7 + (3 + 2)' or '(7 + 3) + 2' can be generated from (a) or (b). Having decided that '7 + (3 + 2)' can be generated from (b) by substituting '7' for '$\underline{\quad}$', '3' for '- - -', and '2' for '$\backsim\backsim\backsim$', we must then decide whether or not '(7 + 3) + 2' can be generated from (a) by these <u>same</u> substitutions.

Further practice of this kind occurs in Unit 2 on page 2-21. These exercises have often proved difficult for students (and sometimes for teachers).

Let's examine the first of the twenty-four expressions at the bottom of the page. We must make fifteen decisions about '9 + 5·6'. [So, in doing this work the student must make 24·15 (that is, 360) decisions. Thank heavens, many of these are very simple.] So, here we are with:

$$9 + 5 \cdot 6$$

The principal operator is '+'. So, '9 + 5 · 6 can only fit a pattern in which the principal operator is '+' or one in which there is no principal operator.

[A single pronumeral is a pattern expression without a principal operator. In this exercise, we are given no patterns consisting of a single pronumeral.] We immediately discard patterns (b), (f), (h), (j), (m), (n), and (o). Now let's examine the ones that remain.

(a)	x + y $\triangle + \square$ 9 + 5 · 6	Yes. Substitute '9' for 'x' and '5 · 6' for 'y'. It may help to use '____' and '_ _ _' or frames at the beginning.
(c)	y + x $\triangle + \square$ 9 + 5 · 6	Yes. Substitute '9' for 'y' and '5 · 6' for 'x'.
(d)	xy + z — · ~~~ + - - - 9 + 5 · 6	No. Remember that we must substitute for each pronumeral in the pattern. We can substitute '5 · 6' for 'z', but without changing '9' to an equivalent expression (which is not permissible in this exercise), we can find no substitutions for 'x' and 'y'.
(e)	a + bc 9 + 5 · 6	Yes. Substitute '9' for 'a', '5' for 'b', and '6' for 'c'. Notice that after matching the **principal** operators we have two new problems. Can you generate the expression '9' from the expression 'a'? Can you generate '5 · 6' from 'bc'?
(g)	ab + ac 9 + 5 · 6	No. See (d) above.
(i)	uw + vw 9 + 5 · 6	No. See (d) above.
(k)	x + (y + z) 9 + 5 · 6	No. We can generate '9' from 'x'. We cannot generate '5 · 6' from 'y + z' because the principal operators are not the same.
( )	(a + b) + c 9 + 5 · 6	No. See (d) above.

Now let's see from which of the fifteen pattern expressions the thirteenth expression at the bottom of page 2-21 can be generated. First, take a good look at the expression:

$$(2a + 3b)5c.$$

If any pattern expression is going to generate this expression, we must be able to "match" the principal operators. Let's unabbreviate.

$$(2a + 2b)5c = (2a + 3b) \cdot 5 \cdot c$$

$$= [(2a + 3b) \cdot 5] \cdot c$$

So the principal operator is the second '$\cdot$'. It follows that this expression can only be generated from a pattern expression whose principal operator is '$\cdot$'. Let's look at them.

(b) $\overset{a}{\triangle} \cdot \overset{b}{\square}$

$[(2a + 3b) \cdot 5 \cdot c$

Yes. Substitute 'c' for 'b' and '$[(2a + 3b) \cdot 5]$' for 'a'.

(f) $x \cdot (y + z)$

$[(2a + 3b) \cdot 5] \cdot c$

No. We cannot find substitutions for 'y' and 'z'.

(h) $(m + n) \cdot p$

$[(2a + 3b) \cdot 5] \cdot c$

No. We can substitute 'c' for 'p'. However, when we try to generate '$(2a + 3b) \cdot 5$' from '$m + n$' we find that the principal operators do not match.

(j) $P \cdot (Q \cdot R)$

$[(2a + 3b) \cdot 5] \cdot c$

No. We cannot find substitutions for 'Q' and 'R'.

(o) $[x \cdot y] \cdot z$

$[(2a + 5b) \cdot 5] \cdot c$

Yes. Substitute 'c' for 'z', '5' for 'y', and '$(2a + 5b)$' for 'x'.

The process by which one determines if a given expression can be generated from a certain pattern expression is purely mechanical. No principles for real numbers are involved.--A. H.

PROVING THE DIVISION THEOREM

Proving the division theorem, pages 2-89 and 2-90, in Unit 2 may be difficult for some students. Mr. Jones and Mr. Edwards of Zabranchburg Junior High School had some ideas about this development. We thought you might find it interesting. If you duplicate and use it, we would like to know how it goes.

Mr. Edwards Teaches the Zabranchburg High Math Class Again !

Remember Mr. Jones who teaches mathematics in Zabranchburg Junior High School and the principal, Mr. Edwards? Mr. Edwards took the class when Mr. Jones was ill. Mr. Edwards had not gone to the same college that Mr. Jones had. However, he knew that the class was studying basic principles and proving theorems. He couldn't stay in the room all hour but he wrote some homework, had his secretary duplicate it and told the students that they could have the hour to do the work. This is a copy of the assignment sheet.

Basic Principles:

(1) $\forall_a \forall_b \; a + b = b + a$ [cpa]

(2) $\forall_a \forall_b \; a \cdot b = b \cdot a$ [cpm]

(3) $\forall_a \forall_b \forall_c \; a + b + c = a + (b + c)$ [apa]

(4) $\forall_a \forall_b \forall_c \; abc = a \cdot (bc)$ [apm]

(5) $\forall_a \forall_b \forall_c \; (a + b) \cdot c = ac + bc$ [dpma]

(6) $\forall_a \; a + 0 = a$ [pa0]

(7) $\forall_a \; a \cdot 1 = a$ [pm1]

(8) $\forall_a \forall_b \; (a - b) + b = a$ [pr]

(9) $\forall_a \forall_{b \neq 0} \; (a \div b) \cdot b = a$ [pq]

Assignment. Using only these nine basic principles, and theorems you prove using them, prove:

$$\forall_x \forall_y \forall_z \text{ if } z + y = x, \text{ then } z = x - y.$$

* * *

Even Fred, "The Brain", was puzzled. He knew what to do if he could use the po and the ps. But Fred knew this wouldn't do because Mr. Edwards said use only the basic principles he had given them and theorems they could prove using these principles. Suddenly Fred thought--just maybe he could

prove a theorem that would take the place of the po. He decided that 'pr' was an abbreviation for 'principle of remainders' and that he would have to use that.

Just what is the pr? It is given this way:

$$\forall_a \forall_b \ (a - b) + b = a$$

Now what does that mean? Fred decided to look at some instances of it. He can get an instance by replacing 'a' by the name of any real number he decides to choose. So he got:

$$\forall_b \ (1 - b) + b = 1$$

$$\forall_b \ (8 - b) + b = 8$$

$$\forall_b \ (-2 - b) + b = -2$$

$$\forall_b \ (0 - b) + b = 0$$

Suddenly Fred said, "Oh". That last instance looked mighty good. He decided he had found something he could use instead of the po, because

if $\quad \forall_b \ (0 - b) + b = 0$

then $\quad \forall_b \ b + (0 - b) = 0 \qquad$ [cpa],

so anytime he wanted to use '$-b$' he could use '$0 - b$' in place of '$-b$'. He decided to call this theorem the 'ot' [Opposites Theorem].

Fred thought things were going pretty well, but he still was a little worried. Maybe he had better see just what he could do with the Opposites Theorem. What would he try it on? Well, the cancellation theorem for addition was used a lot. Maybe he could prove it using his new theorem. Just how would he prove:

$$\forall_a \forall_b \forall_c \text{ if } a + b = c + b, \text{ then } a = c.$$

He thought about the way he would do it if he had the po. [Think about it].

Suppose that $a + b = c + b$.

$(a + b) + (0 - b) = (c + b) + (0 - b)$	[upa]
$a + [b + (0 - b)] = c + [b + (0 - b)]$	[apa]
$a + 0 = c + 0$	[ot]
$a = c$	[pa0]

Hence, if $a + b = c + b$, then $a = c$.

Fred was pleased. He could use his ot instead of using the old po. He decided to tackle the assignment. Prove:

$$\forall_x \forall_y \forall_z \text{ if } z + y = x, \text{ then } z = x - y.$$

"OK", Fred thought, "here goes".

Suppose that (1) $z + y = x$

Fred decided he had to get a subtraction sign into the proof. The only generalization that had a '$-$' in it was the pr.

(2) $$\forall_x \forall_y \ (x - y) + y = x$$

He sat and stared at (1) and (2). Then he realized that from the two of them, he could substitute '$(x - y) + y$' for 'x' and get:

$$z + y = (x - y) + y.$$

Since he had already proved the cancellation theorem for addition, he could get:

$$z = (x - y).$$

So, if $z + y = x$ then $z = x - y$.

He decided that Mr. Edwards knew some mathematics after all. Just then the bell rang.

Mr. Jones didn't have a chance to see Mr. Edwards before the next class meeting. When the class started, Mr. Jones told the students he wanted to introduce the new work on division first so that they could be sure to have enough time to set the stage for the next day's assignment. Then they'd have a discussion at the end of the period. Mr. Jones wrote the Principle of Quotients on the board.

$$\forall_a \forall_{b \neq 0} \ (a \div b) \cdot b = a$$

Then he asked them for some instances of it and wrote them on the board.

$$(2 \div 3) \cdot 3 = 2$$

$$(\tfrac{1}{2} \div 8) \cdot 8 = \tfrac{1}{2}$$

$$(0 \div 2) \cdot 2 = 0$$

$$(1 \div 7) \cdot 7 = 1$$

$$(-5 \div 3) \cdot 3 = -5$$

$$(4 \div -2) \cdot -2 = 4$$

$$(-6 \div -7) \cdot -7 = -6$$

About this time Fred started digging for his scratch paper from the previous day. Where were his instances of Mr. Edwards' pr? Here they were:

$$\forall_b\ (1 - b) + b = 1$$

$$\forall_b\ (8 - b) + b = 8$$

My, but these were like the ones on the board. He wondered if the succeeding proofs would also show a similarity. Just then Mr. Jones said, "Can anyone think of a generalization we might be able to prove immediately by using the pq?"

Fred had one. [What do you think it was?]

$$\forall_{b \neq 0}\ (1 \div b) \cdot b = 1$$

Even Sammy thought that was pretty cool because it was just an instance of the pq.

Mr. Jones asked for another generalization. By now Fred was sure that the cancellation theorem for multiplication could be proved using the pq.

$$\forall_a\ \forall_{b \neq 0}\ \forall_c\ \text{if } ab = cb, \text{ then } a = c$$

[How would you prove it? Hint: Go back to Fred's proof of the cancellation theorem for addition. What will replace '0 - b'? '+'? '-'?]

The assignment for the next day was to prove:

$$\forall_x\ \forall_{y \neq 0}\ \forall_z\ \text{if } z \cdot y = x, \text{ then } z = x \div y\ (= \tfrac{x}{y}).$$

Fred decided this assignment wouldn't be very hard. --A. H.

$\sqrt{8}$: RATIONAL OR IRRATIONAL?

Students may have some difficulty understanding the proof that $\sqrt{8}$ is irrational unless there has been some special preparation. Here is a suggestion for such a development. We'll use .52 as an example.

Teacher: You have said that .52 is a rational number. Can you justify that statement?

Student: Yes. If there are two integers whose quotient is .52 then .52 is rational. One name for .52 is '$\frac{52}{100}$' or '52 ÷ 100'. So there are two integers, 52 and 100, whose quotient is .52.

.52
$\frac{52}{100}$ 52 ÷ 100

Teacher: Can you give me another name for .52, in which the **principal** operator is '÷'?

Student: $\frac{26}{50}$, 26 ÷ 50.

Teacher: Another one. Another one. Another one.

.52
52 ÷ 100
26 ÷ 50
13 ÷ 25
104 ÷ 200
208 ÷ 400
−52 ÷ −100
−26 ÷ −50

Teacher: Let's throw out all negative divisors. Can you fill in the blanks so we will have more numerals for .52?

$$
\begin{array}{c}
.52 \\
\hline
52 \div 100 \\
26 \div 50 \\
13 \div 25 \\
104 \div 200 \\
208 \div 400 \\
\cancel{-52 \div -100} \\
\cancel{-26 \div -50} \\
\underline{\qquad} \div 75 \\
\underline{\qquad} \div 125 \\
\underline{\qquad} \div 500 \\
\underline{\qquad} \div 10
\end{array}
$$

Student: 39, 65, 260

Teacher: How about ______ ÷ 10.

Student: Well, $5\frac{1}{5} \div 10$ is .52, but $5\frac{1}{5}$ is not an integer.

Teacher: How many more numerals like these could we write?

Student: Many.

Teacher: Now really stretch your imaginations. Think of all the possible numerals for .52 that are of this kind. Remember: throw out all negative divisors. Now think of all the divisors.

$$
\begin{array}{rcl}
\multicolumn{3}{c}{.52} \\
\hline
52 & \div & 100 \\
26 & \div & 50 \\
13 & \div & 25 \\
104 & \div & 200 \\
208 & \div & 400 \\
\cancel{-52} & \cancel{\div} & \cancel{-100} \\
\cancel{-26} & \cancel{\div} & \cancel{-50} \\
39 & \div & 75 \\
65 & \div & 125 \\
260 & \div & 500 \\
 & \cancel{\div} & \cancel{10} \\
 & & \vdots
\end{array}
$$

Teacher: How many numbers are there in this set?

Student: Lots and lots.

Teacher: What is the largest number in that set?

Student: 500

Teacher: Stretch your imagination some more--

Student: There isn't a largest.

Teacher: What is the smallest number in that set?

Student: 25

Teacher: You mean that no one can find a name of this type for .52 such that the principal operator is '÷' and the divisor is less than 25?

Student: That's right!

Teacher: Now, let's fill in these blanks.

$.52 \times 100 =$ ______

$.52 \times 50 =$ ______

$.52 \times 25 =$ ______

$.52 \times 200 =$ ______

$.52 \times 400 =$ ______

$.52 \times 75 =$ ______

$.52 \times 125 =$ ______

$.52 \times 500 =$ ______

Teacher: What kind of number did you get in each case?

Student: Real number.

Teacher: That's correct. What kind of real number?

Student: Positive.

Teacher: Right. What kind of positive real number?

Student: Rational.

Teacher: Right. What kind of positive rational number?

Student: Integer.

[We hope it won't take this long to get this answer.]

Teacher: Do you think you can find a positive integer smaller than 25 such that if we multiply .52 by that integer the product will be an integer? How many positive integers are there that are smaller than 25?

Student: 24

Teacher: Let's try them. Mary, you multiply .52 by 1. John, multiply .52 by 2. Kate, multiply .52 by 3. ---Fred, multiply .52 × 24. Did anyone get an integer for the product?

Student: No.

Teacher: Then,

25 IS THE SMALLEST POSITIVE INTEGER SUCH THAT THE PRODUCT OF .52 BY THAT INTEGER IS AN INTEGER.

Teacher: Do you think that we could do the same sort of thing for .125?

Student: Yes.

Teacher: How about $.\overline{3}$? $\frac{25}{30}$? $.\overline{6}$? $\frac{2}{3}$? Do you believe we could do this for any rational number?

Student: Yes.

When you are ready for page 4-48, you might do something like:

Teacher: Let's think about $\sqrt{8}$. The text tells us [bottom of page 4-47] that $\sqrt{8}$ is not rational. Let's prove it. Suppose it were rational. Now if it's really not rational and we suppose it is what is going to happen?

Student: We'll be in a mess.

Student: Everything will get fouled up.

These answers were actually given in a class where this was tried.

Teacher: Right.

PROVE $\sqrt{8}$ IS NOT RATIONAL SUPPOSE $\sqrt{8}$ IS RATIONAL

[Now quickly review the development that led to

25 is the smallest positive integer such that the product of .52 by that integer is an integer.]

Teacher: If $\sqrt{8}$ is rational can you make a sentence like this one?

Student: Yes.

Teacher: What marks in this sentence will have to be replaced?

Student: '25' and '.52'.

Teacher: What shall we put in place of '.52'?

Student: '$\sqrt{8}$'.

Teacher: Do you know what to put in place of '25'?

Student: No.

Teacher: Let's use a pronumeral, 'q'.

Let's call that product integer 'p'.

> q IS THE SMALLEST POSITIVE INTEGER
> SUCH THAT $\sqrt{8} \cdot q$ IS AN INTEGER
> So, $\frac{p}{q}$ = ______ .

How shall I fill in the blank?

Student: '$\sqrt{8}$'.

[Now, you are ready for line 5 on page 4-48.]

Teacher: I am thinking of a number that is a positive integer and it is less than q. Let's call it 'r'. What can you tell me about r? Let's call it 'r'.

Student: It's rational.

Teacher: That's true. What else? [If necessary, say]. Look at this sentence on the board.

> q IS THE SMALLEST POSITIVE
> INTEGER SUCH THAT $\sqrt{8} \cdot q$ IS
> AN INTEGER.

What can you tell me about $\sqrt{8} \cdot r$?

Student: $\sqrt{8} \cdot r$ is not an integer.

Teacher: Right! If you multiply $\sqrt{8}$ by a positive integer which is less than q the product is not an integer. Let's put that up here.

> q IS THE SMALLEST POSITIVE INTEGER
> SUCH THAT $\sqrt{8} \cdot q$ IS AN INTEGER.
> $\frac{p}{q} = \sqrt{8}$
> $\sqrt{8}$ $\cdot$(ANY POSITIVE INTEGER LESS THAN q) IS NOT AN INTEGER

Now, let's think about $\sqrt{8}$. Can you tell me an integer that is larger than $\sqrt{8}$?

Student: 10, 9, 1,000,000, 3.

Teacher: One that is smaller than $\sqrt{8}$?

Student: $-10, -1{,}000{,}000, 0, 1, 2.$

Teacher: Let's use 2 and 3. So

$$2 < \sqrt{8} < 3$$

Or if we use '$\frac{p}{q}$' instead of '$\sqrt{8}$' we write '$2 < \frac{p}{q} < 3$'.

Now let's use the transformation principles and get some equivalent inequations.

Student: $2q < p < 3q$ [$p > 0$ and the multiplication principle for inequations.]

Teacher: Can you get another one that begins

$$0 <$$

Student: $0 < (p - 2q) < q$

Teacher: So what do we know about $p - 2q$? Can it be 0? Can it be negative? Can it be as large as q?

Student: $(p - 2q)$ is positive and is less than q.

Teacher: What kind of number is $(p - 2q)$?

Student: Integer. The set of integers is closed under multiplication and subtraction.

Teacher: So $(p - 2q)$ is a positive integer that is less than q. Now finish this sentence:

$$\sqrt{8} \cdot (p - 2q) \text{ is } \underline{\hspace{3cm}}$$

Student: Not an integer!

Teacher: Let's use '$\frac{p}{q}$' instead of '$\sqrt{8}$'.

$\boxed{\frac{p}{q} \cdot (p-2q) \text{ IS NOT AN INTEGER}}$

Let's write this in some other form. What expression is equivalent to '$\frac{p}{q} \cdot (p - 2q)$'?

Student: $\frac{p^2}{q} - 2p$

$\boxed{\left(\frac{p^2}{q} - 2p\right) \text{ IS NOT AN INTEGER}}$

Teacher: Let's get really clever now and find another expression that is equivalent to

$$\frac{p^2}{q}$$

What is one principle that is very useful in getting an equivalent fraction?

Student: Principle of mulitplying by 1.

Teacher: Now think of expressions that are equivalent to 1 and we'll try some of them in the blank.

$$\frac{p^2}{q}, \underline{\qquad\qquad} - 2p \text{ is not an integer}$$

Student: $\frac{'p'}{p}, \frac{'2'}{2}, \frac{'q'}{q}$

Teacher: Let's use that last one.

$$(\frac{p^2}{q} \cdot \frac{q}{q} - 2p) \text{ is not an integer}$$

Student: So $[(\frac{p^2 q}{q^2} - 2p)]$ is not an integer

Teacher: I rather like that. Look at this part of it.

$$\boxed{\left(\left(\frac{p^2}{q^2}\right)q - 2p\right) \text{ IS NOT AN INTEGER}}$$

How else could this be written?

Student: $(\frac{p}{q})^2$

Teacher: [So $(\frac{p}{q})^2$ q - 2p] is not an integer

Say, just what was $\frac{p}{q}$ anyway?

Student: $\sqrt{8}$

Teacher: So,

$$\boxed{[(\sqrt{8})^2 \cdot q - 2p] \text{ IS NOT AN INTEGER}}$$

Student: But, ---but---but $(\sqrt{8})^2$ is 8 and 8's an integer. So 8q is an integer and 2q is an integer. So, (8q - 2) is an integer.

Teacher: How about that? What's happened?

Student: We've made a mistake?

Teacher: Let's see if we did. What was our original problem?

Student: Show $\sqrt{8}$ is not rational

Teacher: How were we going to do that?

Student: By supposing $\sqrt{8}$ were rational and that that would foul things up.

Teacher: Well, has that supposition "fouled things up"?

Student: Yes.

Teacher: Let's stop calling this a "mess" and say that:
"Our supposition has led to two contradictory statements".
Now what do you conclude?

Student: That $\sqrt{8}$ is not rational.

Show $\sqrt{8}$ is not rational

Suppose $\sqrt{8}$ is rational

q is the smallest positive integer such that $\sqrt{8} \cdot q$ is an integer (p).
So $\frac{p}{q} = \sqrt{8}$

$\sqrt{8} \cdot$ (any positive integer less than q) is not an integer

$2 < \sqrt{8} < 3$

$2 < \frac{p}{q} < 3$

$2q < p < 3q$

$0 < (p - 2q) < q$

So $(p - 2q)$ is a positive integer less than q.

Hence $\sqrt{8}\,(p - 2q)$ is not an integer

$\frac{p}{q} \cdot (p - 2q)$ is not an integer

$\left(\frac{p^2}{q} - 2p\right)$ is not an integer

$\left(\frac{p^2}{q} \cdot \frac{q}{q} - 2p\right)$ is not an integer

$\left(\frac{p^2 q}{q^2} - 2p\right)$ is not an integer

$[(\frac{p}{q})^2 \cdot q - 2p]$ is not an integer

$[(\sqrt{8})^2 \cdot q - 2p]$ is not an integer

$\left.\begin{array}{r}(8q - 2p) \text{ is not an integer} \\ \text{but it is an integer}\end{array}\right\}$ Contradiction

So $\sqrt{8}$ is not rational. --A. H.

A TEACHER'S VIEW OF UICSM

(This article is a slightly edited version of the speech given by Mr. Howard Marston of the Principia Upper School St. Louis, Missouri, as his contribution to the panel of teachers who commented upon their experiences with the several new school mathematics programs at the Topeka, Kansas, Regional Orientation Conference of administrators and supervisors.)

I think it should be made clear that these various programs in developing mathematics curricula are not in competition with each other. Each is organized in a different way, and each serves a somewhat different purpose.

We are presently in our sixth year of teaching the UICSM materials at Principia in St. Louis. The UICSM program had its beginnings in 1951, and in 1955 we entered the program as the fifth school to join as a "pilot" school. There are now 144 cooperating schools.

We had heard a talk by one of the staff telling what they were doing at Illinois; we wrote them letters and talked with them until we were finally accepted as one of the schools in which to try their newly written materials. Our mathematics department wanted to try out the materials, and so did the administration. (It is interesting to note that both our headmaster and assistant headmaster were former math teachers.) We were all of the opinion that the students couldn't learn any less mathematics than they were learning in the traditional classes, and perhaps they would learn more.

We began that fall of '55 by having another teacher and myself each teach one 9th grade UICSM class. We had had no special training for teaching these materials. I myself had taught conventional mathematics at public and private schools in the East for five years prior to this, and the other teacher had taught mathematics for many years.

Not only did we not have special training at the time, but we did not even have all the materials. The materials, in mimeographed form, were being mailed to us almost on a day-to-day basis at times, and usually the commentary for teachers arrived too late. Max Beberman and David Page made several visits to our school during the year and we visited them in Urbana three or four times to receive the latest information and instruction. Of course, we kept in contact by mail and sometimes in emergencies by telephone. It was not long before we were completely sold on the program despite those early inconveniences, and so were our students.

What we liked most was the fact that the students had to think and, although this was a new experience for them, they began to enjoy it. Oh, there were one or two students who said "Just give me the rules and I'll do the work." But most students soon entered into the spirit of the game. We discourage students from helping one another (we like to let each student discover his own rules and short cuts through careful development of the exercises), and we discourage the students from getting help from their parents. It isn't long before the parents are "lost" anyway.

This brings up the point of parent reaction to the program. Every so often we try to explain in simple terms at parent's meetings what we are trying to do. The vast majority of parents have been most cooperative and understanding. I have been very pleased in this respect, and so, I believe, has our administration. The best ambassadors to the home front have been the students themselves. From the "A" student on down to the "just passing"

student, there is much enthusiasm. This is due in part to the freshness with which the materials are written, and in part to the fact that through the way in which the students discover much of the material, they feel it is theirs rather than something that is being forced on them. Far less frequently, practically not at all, do we get the question "Why do we have to know this?" Students love to be creative. The better students particularly enjoy the more challenging concepts.

How do the students do on the standard tests? We do not have much in the way of statistics, but we believe that on the Cooperative tests of the Educational Testing Service our UICSM trained students do as well as the traditionally taught. In the next year or so, as the tests are revised to incorporate more of the newer topics, we feel our students will do much better. The present tests do not test the students over all the material with which they are familiar. Nor do they do much in testing for understanding. As for College Board tests, we had very fine scores this last year--higher than ever before. About 95% of our students are college-bound, but their ability is not exceptional. We get some very excellent scholars, but we have some students who have not shown themselves to be strong students but who go to college perhaps because their parents want them to. You see, two-thirds of our high school population (boys and girls) consists of boarders whose parents have the means to send them to such a school. Our scores on the advanced mathematics achievement test of the College Board were higher this year than ever before. For the first time we had one student receive the top score of 800, while 5 out of 9 received scores of 720 or better. This class had some exceptional students in it, so I do not pretend that the UICSM program should take all the credit, but it certainly did them no harm.

One of the underlying principles of the UICSM program is that students are capable of better thinking than teachers have given them credit for; this is in line with the philosophy of our school: that we do not put limitations on the mental capacities of the students.

Another point we like about the program is the precision of language and the use of elementary logic in the texts. The teacher is expected to use this same care in language, and, although the students aren't expected to be so careful themselves, they do grow to use care and to appreciate doing so. I have had students tell me that they are particularly pleased with this training, and that, in becoming aware of it, they are more precise and less ambiguous in all that they say and write.

We had such confidence in the program that after three years we eliminated our first year algebra course; all students, instead, begin their high school mathematics with UICSM First Course. We have one exception to this: namely, one class of the top eighth grade students at our school which begins First Course in the eighth grade.

We have found this program to be so successful that this year our classes from first grade through sixth grade are all using materials from David Page's University of Illinois Arithmetic Project. This program is entirely separate from the UICSM program, but is an outgrowth of it.

It so happens that there is a comparatively large turnover of our students each year, so it is difficult to say whether students continue with mathematics longer than those with conventional backgrounds. We like to think that

they do. But this transfer of students raises another problem: how do our students fare who transfer to other school systems after a year or two? From what we hear, they have no difficulty. After all, after the first year they have acquired most of the skills conventional algebra students have learned, and following the second year they have learned the Euclidean geometry. Even after three years they have learned the skills taught in intermediate algebra, except that I find that my classes cannot reach the study of complex numbers until the senior year.

The real problem is the student who transfers to our school from other school systems. We have kept conventional classes in the 10th, 11th, and 12th grades for these students. The first few years we integrated a few of these transfer students who wanted to study, or whose parents wanted them to study, the UICSM math. This was not easy to do, and, as the UICSM took bolder steps in revising their program, we stopped doing this altogether.

This year, though, we are trying something new. We have taken the best of those 10th grade students who had a year of conventional algebra at other schools and formed a special class in mathematics for them. To these students, from September until now, I have been teaching selected topics from the first four units, which constitute a year and a third of the UICSM program at our school. These are topics they did not have or had very little of in algebra. Now for the rest of the year this class will study the UICSM Second Course. Next year they can be integrated with our other Third Course students.

You should have seen the skeptical looks on the faces of these students the first week as they saw they were studying ninth grade material; since then they have become so excited about their math that there are times when they can hardly stay in their seats. One girl after two weeks said: "I used to hate math. The teacher would do some problems and then we would do more like them for homework. It was boring. But I just love this class. I really have to think." Others have decided now to major in mathematics. The parents of several of these students have expressed appreciation for the program because of the enthusiasm of their children for it.

What about those students who have had four years of the UICSM math and have gone on to college? We have two such groups. These students report that they are being exempted from certain math courses and are in some instances being allowed to study honors courses. We teach our students no calculus. I would not call our course an accelerated course: it is more enrichment, giving the students solid background and real understanding. I asked one of my former students how well she thought she was prepared for her college math--having studied the UICSM course--compared to her classmates at college. She replied, "Better than any of them." The class was using Taylor's analytic geometry and calculus text. She said that there were so many questions about proving this and deriving that which the other students found quite difficult, but which caused her no trouble at all since she had been doing just that for four years in high school.

The teacher has an extremely important part in the success of a program. The teacher's role is a bigger one in the UICSM classroom than in a conventional classroom--or in an SMSG classroom, as Dr. Price mentioned earlier--because of the nature of the way in which concepts are carefully

developed. It takes more work and harder study for the teacher to prepare for classes. I feel it is vitally important that before a teacher teach these materials he have had a summer's training in them.

I have had the good fortune to help train some of these teachers. For the last three summers I have been an instructor at the National Science Foundation Institute at the University of Arizona in Tucson. I have taught First and Second Course of UICSM to teachers there and Second Course to teachers at the University of Illinois one summer. I have been interested in their reactions. These teachers are interested in what the program is all about, but many are at first skeptical and quite disturbed as they see the weaknesses in the way they have been teaching, some of them for many years. At first they seem to resent the fact that their explanations (and those of their textbooks) have been insufficient and not sound mathematics. Then they have become really appreciative of what they are learning. Toward the middle of the summer as we study Units 3 and 4 they say that they would like to go back to Unit 1 to study it with real appreciation for what it is doing and the excellent ground work it is laying. A teacher's just reading the units is not enough. He has to see it in action and even try teaching it for one or two years before he can really appreciate the beautiful continuity and the wonderfully clear development of concepts.

Last summer my teacher-students who were studying Unit 5 on relations and functions and who had studied First Course the year before found Unit 5 so exciting they neglected their other courses to read it. They nicknamed the unit the "Beberman novel" because they just couldn't put it down. These teachers have told me that, whether their school systems would let them join the program or not, their study of these materials certainly affected their teaching in their conventional classes.

Unit 6 on geometry teaches geometry and proof with new meaning and significance. I have found it extremely enlightening.

If you listened to last Tuesday morning's television showing of the series "Continental Classroom: A Course in Modern Algebra" you heard the speaker mention some interesting facts about how the UICSM program teaches mathematical induction to its 11th grade students. In most algebra books--high school and college--the topic of mathematical induction is a short chapter in which one gets the idea that to prove a generalization he must assume the thing he is trying to prove. How confusing! Our Unit 7 carefully develops the idea of mathematical induction and allows the students to really understand what mathematical induction is all about. We teach mathematical induction because we use it directly, and indirectly, to prove our theorems about exponents and also theorems for arithmetic and geometric progressions. Ability to manipulate exponents is necessary, but incidental to the careful development of the mathematical concept of exponents. The way in which trigonometry is developed is also extremely clear, enlightening, and sound. As you can see from my comments, I could never go back to teaching mathematics from traditional textbooks. These UICSM materials do as much for the teacher as for the student.

Lest I leave you with the impression that <u>all</u> is "rosy" and that this is a panacea for <u>all</u> the ills of teaching mathematics, let me say that this is not the case. Some students <u>still</u> don't care particularly for math, some

students still have a great deal of trouble with it, some students never learn to manipulate fractions, some students transfer from our UICSM program to our traditional, thinking that the grass is greener in the other yard; but there are not many of these instances, and the good reports far outweigh the bad. Also, some teachers do not see the use of this program, and some outstanding mathematicians do not agree with the philosophy of the program. But it couldn't be much different from the traditional if everyone were in complete agreement about it.

So, in summary, I feel that the UICSM program is a worthy one if for no other reason than that the teachers gain renewed interest in teaching math. But, the students do learn a lot more math, they enjoy it, they understand it, they do as well on tests, they become creative thinkers rather than automatic manipulators, and they enjoy abstractions. The more abstract the mathematics is, the more practical the students tell you it is.

* * *

NEWS AND NOTICES

Sister Mary of the Angels, St. Rosalia High School, Pittsburgh, represented UICSM at the Philadelphia Regional Orientation Conference in October and appeared with Mr. Beberman as a panelist at the annual Edison Foundation Institute in Pittsburgh on November 17 and 18.

Mr. Howard Marston of the Principia Upper School in St. Louis was the teacher-panelist representing UICSM at the Topeka, Kansas, Regional Orientation Conference on December 1 and 2. (This Conference was omitted from the list published in Newsletter No. 2, for which we apologize.)

Mrs. Teruko S. Yamaura of the Kapaa High and Elementary School, Kapaa, Kauai, Hawaii, spoke on the UICSM program before some sixty Rotarians who visited her school during American Education Week. She reports that "It was really a good experience for me, and they showed great interest."

Miss M. Eleanor McCoy, associate teacher coordinator for the project, returned recently from an extended visitation trip to cooperating schools in Hawaii and some continental Western states. There are 1900 students enrolled in UICSM classes in the 24 cooperating schools of our newest state.

Mr. Beberman's travel plans for February include the following appearances and presentations:

February	3	Lyons Township Teacher's Institute, LaGrange Park, Illinois
	8	Address the Principals of District One at Chicago, Illinois
	22	Address the Southern Regional Science Seminar for University Information Officers at the University of Florida, Gainesville
	24, 25	Participant in Conference on Future Responsibilities for School Mathematics in Chicago, Illinois

NOTICE

This issue of the UICSM Newsletter is the last to be mailed to schools for which we have no record of the individuals who receive it. Future issues will be sent to named individuals only, so those who wish to receive further issues but are not named on the address label must send us their name and address. A postal card will suffice.

NOTICE

UICSM will conduct a six-week summer training conference in Urbana from June 26 through August 4, 1961. There will be classes for those preparing to teach any of the four courses in the UICSM program. We have received a grant from the National Science Foundation which will enable us to pay a stipend and allowances to participating teachers.

Administrators of cooperating schools have been sent information on applications, course credit, housing, stipends, and allowances. Teachers in cooperating schools who might be interested in this program should consult with their principal or superintendent about applying. Other teachers using UICSM materials may wish to write to us for information and application forms.

We know of three other institutions with summer offerings which will prepare teachers of UICSM first or second course. For information, write to the NSF Summer Institute Director, c/o the Department of Mathematics, at one of the following:

Sacramento State College
Sacramento 19, California

Wayne State University
Detroit 2, Michigan

Western Washington College of Education
Bellingham, Washington

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OPERATIONS AS FUNCTIONS

Many of the concepts in Unit 5, Relations and Functions, can be readily anticipated in Units 1-4. [And, this can be done without using the words 'relation' or 'function'.] Here are some ideas about how to do this. [We are assuming that the reader is familiar with Unit 5.]

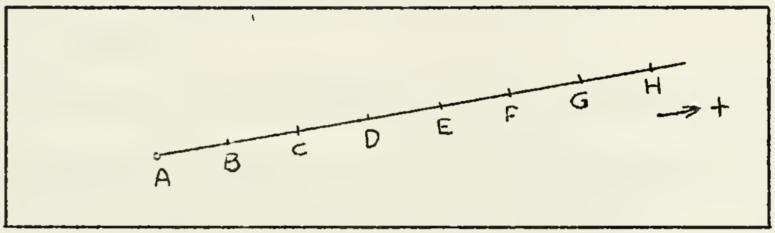
Let's start with Unit 1. Near page 1-7, we can make our first contribution to a better understanding of:

(1) a relation is a set of ordered pairs;

(2) a function is a set of ordered pairs [relation] no two of which have the same first component.

*

Consider a road which begins at A.



Teacher: What is the starting point of a trip whose measure is $^{+}2$ [or, 2 to the right, or $\overrightarrow{2}$]?

Student: A.

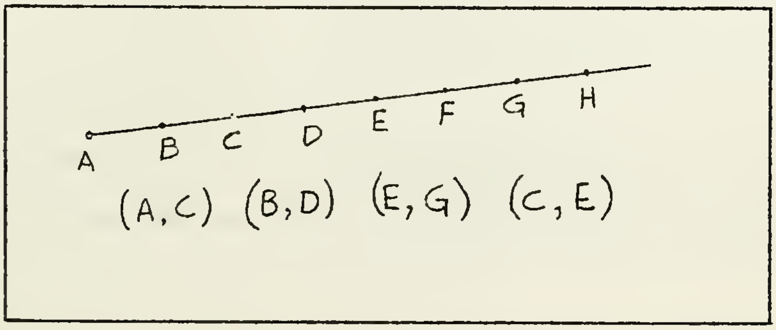
Teacher: What is the ending point?

Student: C.

Teacher: Is the trip from A to C the only trip whose measure is $^{+}2$?

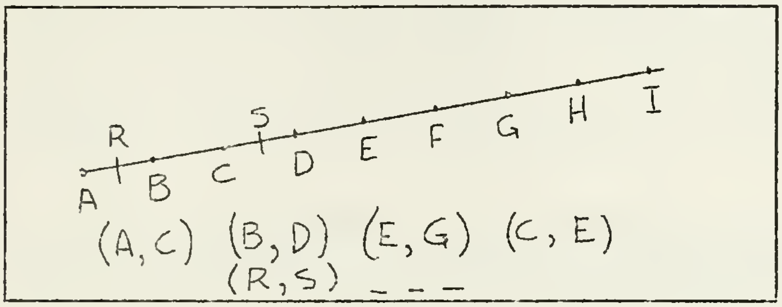
Student: No. The trip could begin at B and end at D.

Teacher: Any other trips of this kind?



Teacher: We're getting a lot of these now. To keep from getting them mixed up, let's use parentheses.

Student: A trip beginning midway between A and B and ending midway between C and D has measure $^{+}2$.

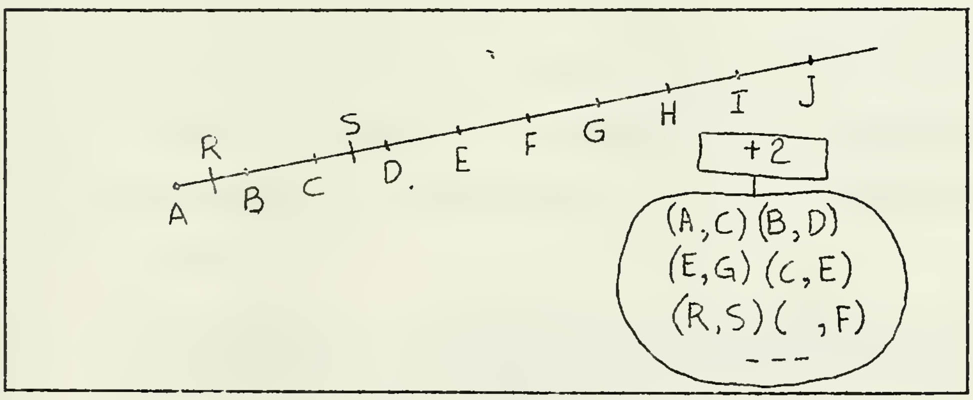


Teacher: How many trips are there with measure $^{+}2$?

Student: Lots of them.

Teacher: So all these trips and many, many more have measure $^{+}2$.

[Note the "balloon" notation introduced here. This anticipates page 1-68. Also, see the discussion on TC[1-1]a and TC[1-1]b.]



Teacher: Is there a trip whose measure is $^{+}2$ and which ends at F?

Student: Yes, the trip from D to F.

Teacher: Does the trip from E to C belong to this set? [The child will understand the word 'set' here without any explanation.]

Student: No.

Teacher: Does the trip beginning at A and ending at D belong to this set?

Student: No.

Teacher: [Pointing to '(A, C)'.] Is there any other trip besides this one that starts at A and has measure $^{+}2$?

*

It is possible that some student may contend that the trip from A to C made today is different from the trip from A to C made yesterday. If this happens make clear that the word 'trip' is being used as the person who says, "I've made the trip from Urbana to Chicago many times", is using it

This development lays the groundwork for defining a real number as a certain set of ordered pairs of numbers of arithmetic. If we consider the road to be a number ray of arithmetic, we have defined $^{+}2$ to be the set of ordered pairs of numbers of arithmetic such that the second component of each ordered pair is 2 greater than its first component.

Now, what can we do with section 1.02, Addition of real numbers?

*

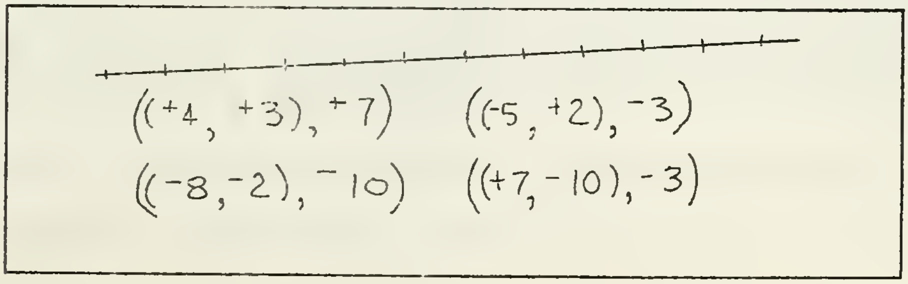
Teacher: Mary, think of a trip on this road. What real number measures the trip you are thinking about?

Student: $^{+}4$.

Teacher: Beginning at the end-point of that trip, take a trip whose measure is $^{+}3$. What is the measure of the single trip that would take you from the beginning point of the first trip to the ending point of the second trip?

As each number is given, write a numeral for it in the proper place, putting in the commas and parentheses as you write. The stages by which you would arrive at '(($^{+}4$, $^{+}3$), $^{+}7$)' are:

$$^{+}4$$
$$(^{+}4,\ ^{+}3)$$
$$((^{+}4,\ ^{+}3),\ ^{+}7)$$



Student: $^{+}7$

Teacher: Correct.

*

Continue the questioning in order to obtain additional ordered pairs whose first components are themselves ordered pairs.

$$((^{+}4,\ ^{+}3),\ ^{+}7),\quad ((^{-}5,\ ^{+}2),\ ^{-}3),\quad ((^{-}8,\ ^{-}2),\ ^{-}10),\quad ((^{+}7,\ ^{-}10),\ ^{-}3)$$

$$((^{+}3,\ ^{+}4),\ ^{+}7)$$

[Naturally, we are not suggesting that this ordered pair notation should replace the conventional '$^{+}4 + {}^{+}3 = {}^{+}7$' type of listing of addition "facts".]

Again we have a set of ordered pairs, no two of which have the same first component. A function of this type in which the first component of each ordered pair is itself an ordered pair is often called <u>a binary operation</u>. Contrast this with:

$$\{(^{+}4,\ ^{+}7),\quad (^{-}3,\ 0),\quad (^{-}5,\ ^{-}2),\quad (^{+}8,\ ^{+}11),\quad (^{-}2,\ ^{+}1),\quad ---\}$$

This operation, adding $^{+}3$, is called <u>a singulary operation</u> on the set of real numbers.

Some mathematicians use the phrase 'operation on a set S' only if the set is closed with respect to the mapping. Thus, a singulary operation on S would be a mapping which takes you from a member of S to one and only one member of S. In the case of a binary operation on a set S, the mapping would take you from any member of $S \times S$ to one and only one member of S. In UICSM courses, we use the word 'operation' as synonymous with 'function'. So, for example, we talk about the absolute value operation which takes you from the real numbers to the numbers of arithmetic. This is not a usage of 'operation' which conforms to the definition mentioned above.

* * *

Part of the following is a modified transcription of the questions and answers given in one of the classes at University High School. Some of you may recall it from one of our training films.

\- - -

The day before this discussion, the class worked exercises designed to create an awareness of the existence of the inverse of an operation. For example, among them were exercises such as "If you want to undo the result of adding 15, subtract ______ from the sum." Now the class is ready to find out that an operation is a set of ordered pairs and then to find how to form the inverse of an operation. The terminology 'converse of an operation' can also be introduced here, but this is optional.

Teacher: Suppose a third-grader said to you, "What do you mean by 'adding 9'?" What would you tell him?

Student: Well---add a number---

Teacher: That wouldn't be very helpful.

Student: Well, adding nine ones to whatever you're adding.

Teacher: I don't know whether that would help him or not.

Student: (Mary) Well, if he had one apple and you gave him nine more apples, he would have ten apples.

Teacher: I see. What would you say, Jack?

Student: Uh, have him a quantity-----so much more.

Teacher: Joan, what do you think?

Student: Well, he ought to know it.

Teacher: Let's see. Mary, you said to give him an example. He has one apple, give him nine apples. He now has ten apples.

$$1 + 9 = 10$$

What else would you do? Give him another example? Harry.

Student: You could say you have 12 apples. Then someone gives you nine more apples. Let him count them up to see how many he has.

$$1 + 9 = 10$$
$$12 + 9 = 21$$

Teacher: You could keep on giving him example after example. I think pretty soon he'd form some idea of what adding 9 is. Let's put some more examples down like these.

1 + 9 = 10
12 + 9 = 21
31 + 9 = 40
7 + 9 = 16
30 + 9 = 39
10 + 9 = 19
200 + 9 = 209
2 + 9 = 11
3 + 9 = 12

Teacher: Imagine, for a moment, that we have all possible examples of adding 9. How many would there be?

Student: Lots of them. But he won't know how to add 8.

Teacher: Then we'll make him a specialist in adding 9. Imagine that we have all possible examples for adding 9. How do we solve a problem in adding 9? Imagine we had a book <u>full</u> of these examples of adding 9. Then we gave him a problem:

6 + 9 = ?

What would he do?

Student: Look in the six's column.

Teacher: Look for the example where a '6' appears in the first column. Suppose he finds it. What would he find in the last column?

Student: 15.

Teacher: So, he knows the answer to this problem must be 15. Let's pretend that we're actually going to have a book like this--all full of examples of adding 9. Now, I want to save some space in the book. Is there any way in which I can shorten these sentences so I can save space?

*

The students made various suggestions about how to shorten these sentences. Their suggestions amounted to something like the following:

"We don't need all those plus signs. Let's take them out." [The plus signs were all erased.] Someone else says, "Why not remove all the equal signs?" [Equal signs were erased.] Finally, someone says, "Why not take out all the '9's, since they are repeated in each sentence?" The teacher then pointed out that they needed something to separate the umerals for each pair; so, commas were decided upon. Finally, the teacher said, "Let's put the whole works in parentheses." The following shows different stages of the development.

1	9	=	10
12	9	=	21
31	9	=	40
7	9	=	16
30	9	=	39
10	9	=	19
2	9	=	11
3	9	=	12
6	9	=	15

1	9	10
12	9	21
31	9	40
7	9	16
30	9	39
10	9	19
2	9	11
3	9	12
6	9	15

1	10
12	21
31	40
7	16
30	39
10	19
2	11
3	12
6	15

1	,	10
12	,	21
31	,	40
7	,	16
30	,	39
10	,	19
2	,	11
3	,	12
6	,	15

(1	,	10)
(12	,	21)
(31	,	40)
(7	,	16)
(30	,	39)
(10	,	19)
(2	,	11)
(3	,	12)
(6	,	16)

Recopy to save space

(1, 10)
(12, 21)
(31, 40)
(7, 16)
(30, 39)
(10, 19)
(2, 11)
(3, 12)
(6, 15)
- - -

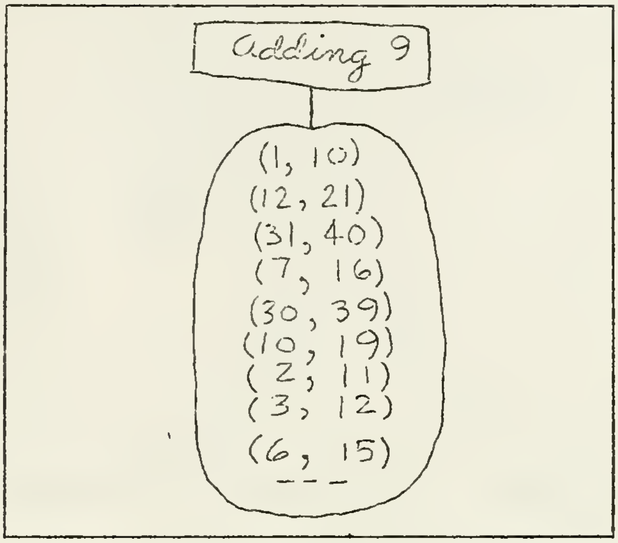
*

Teacher: Imagine then, these are just a few sample pairs from this book. What would be a good name for the book? Suppose we actually got real silly and printed a book like this. What would be a good name for the book? 'David Copperfield'? What would be a good name, John?

Student: Adding Nine.

[Other suggestions.]

Teacher: Let's pick that short title: Adding Nine.



Teacher: This book has lots more pairs. Let's get a few more samples to see that we've got the idea. A pair that begins with 15, ends with what?

Student: 24

Teacher: A pair that begins with 40 ends with what?

Student: 49

Teacher: I'm going to write a pair and I want you to tell me if it really could come from this book. Ready? (17, 26). Yes? No? Class?

Class: Yes

Teacher: (39, 30) Yes? No? Class?

Class: No.

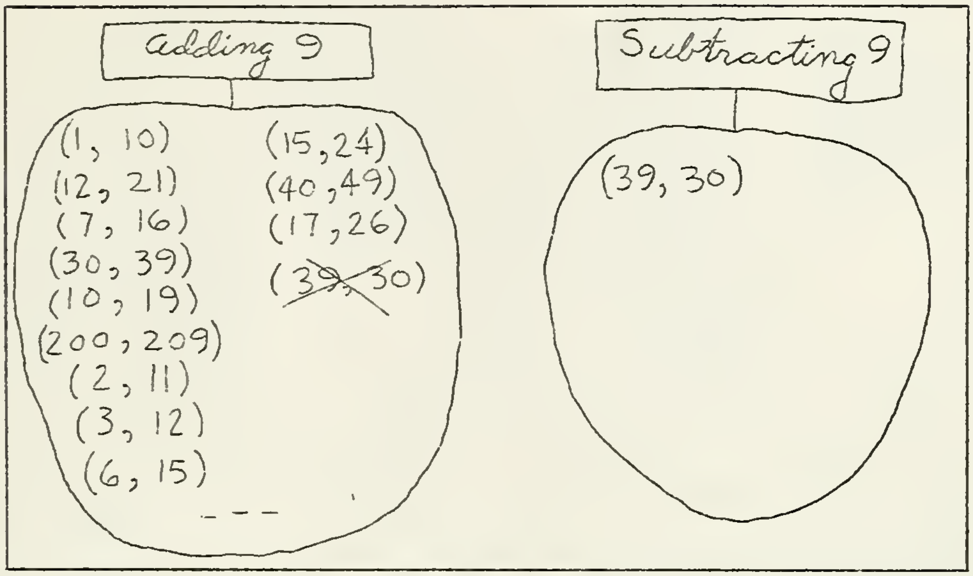
Teacher: What about this pair, (39, 30)? What book could that come from?

Student: The Book for Subtracting Nine.

Teacher: I think I like that.

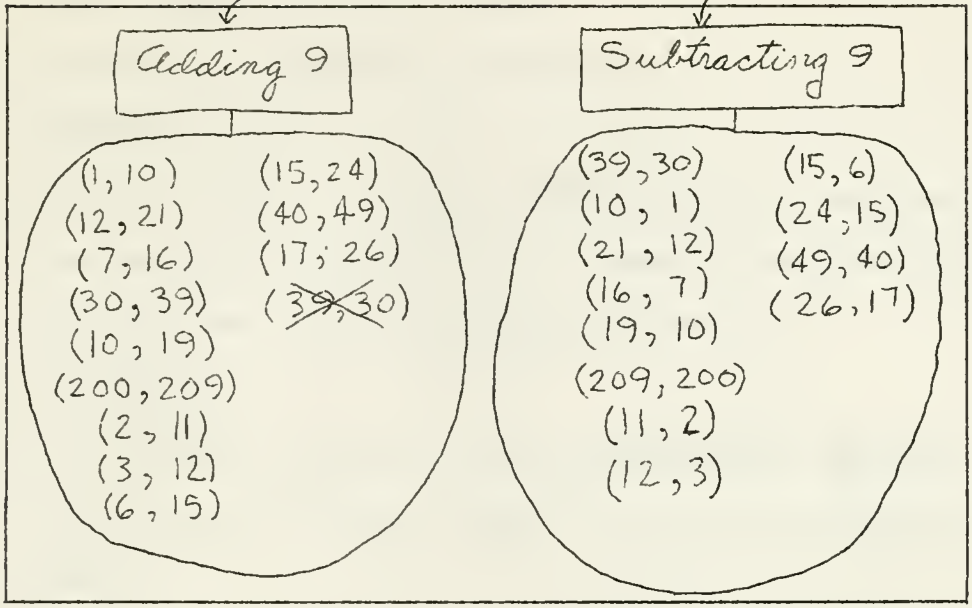
Student: The Book for Adding Negative Nine.

Teacher:	Good, but remember we were talking about numbers of arithmetic. [Third-grade.]



Teacher:	Let's get some more pairs that would belong to the book, Subtracting Nine. I want someone very quickly--without doing much thinking--to give me a whole bunch of pairs that come from this book. Are you ready?
Student:	(69, 60)
Teacher:	Too much thinking.
Student:	(10, 1)
Teacher:	Faster.
Student:	(21, 12), (16, 7), (39, 30).
Teacher:	You already have (39, 30). [(39, 30) was the first entry in this book.]
Student:	(19, 10), (209, 200), (11, 2), (12, 3), (15, 6), (24, 15), (49, 40), (26, 17).
Teacher:	Are there any more pairs that belong to Subtracting Nine?
Student:	Lots.

Teacher: If you have this first book, do you need the second book?



Student: No.

Teacher: Right. It turns out that this second book is not really necessary. Any problem you want to do using the second book could have been done by using the first book.

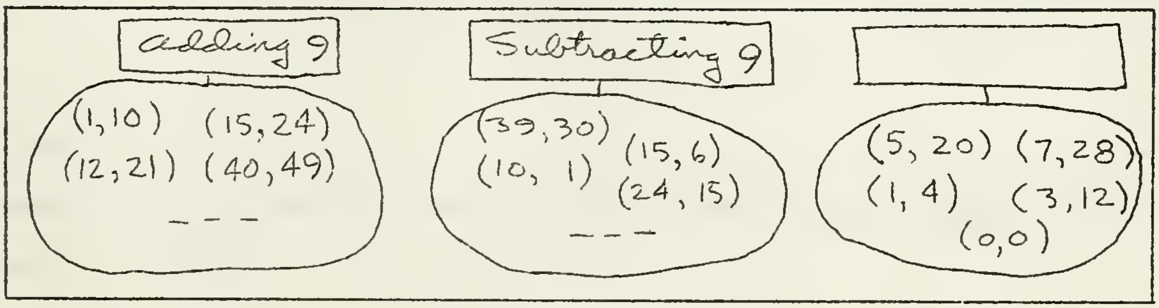
Student: For that matter, you could say that the Adding Nine book was not necessary. You could use the Subtracting Nine Book.

Teacher: Right. But, suppose that we all know how to add 9, does that automatically tell us how to subtract 9?

Student: Yes.

Student: Would there be books like these for multiplying and dividing?

Teacher: Well, I think so. Let's make up another book. Here are some of the pairs that belong to it.



Teacher: Give me some more pairs that belong to this book.

Student: (400, 1600), (80, 320), (9, 36)

Teacher: What is a good name for this book?

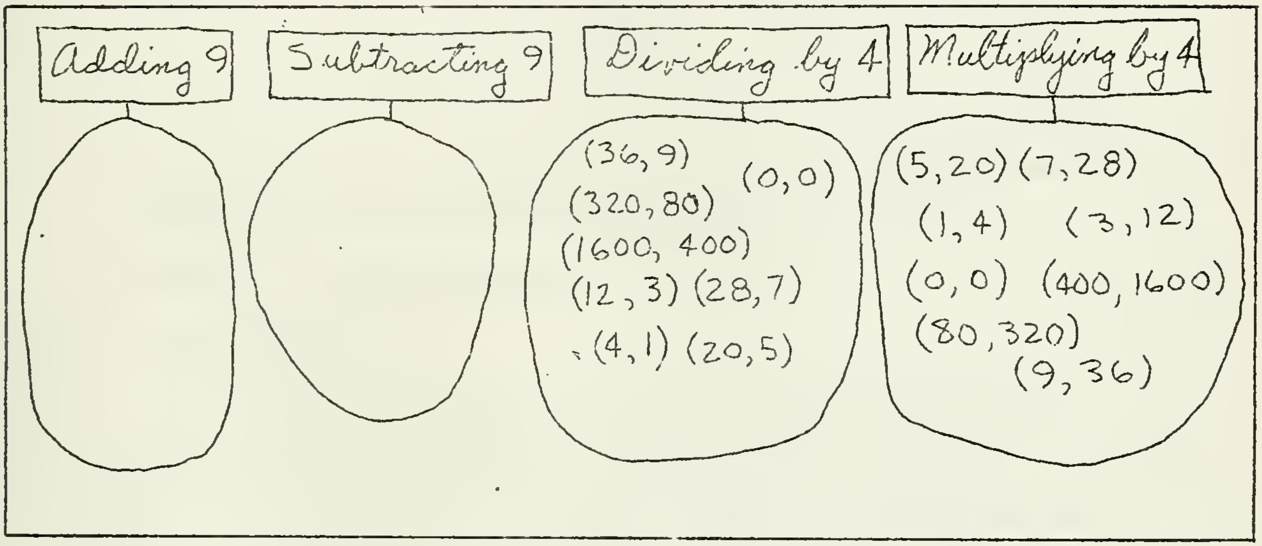
Student: Multiplying by 4.

Teacher: Remember that we could do without the Subtracting Nine book if we had the Adding Nine book? Is there any book we could do without if we had the Multiplying by 4 book?

Student: Dividing by 4.

Teacher: Tell me some of the pairs that would belong to that book.

Student: (36, 9), (320, 80), (1600, 400), (12, 3), (4, 1), (0, 0), (28, 7), (20, 5).



Teacher: Does anyone see how he gets the pairs that belong to this book?

Student: Well, he just flips them around.

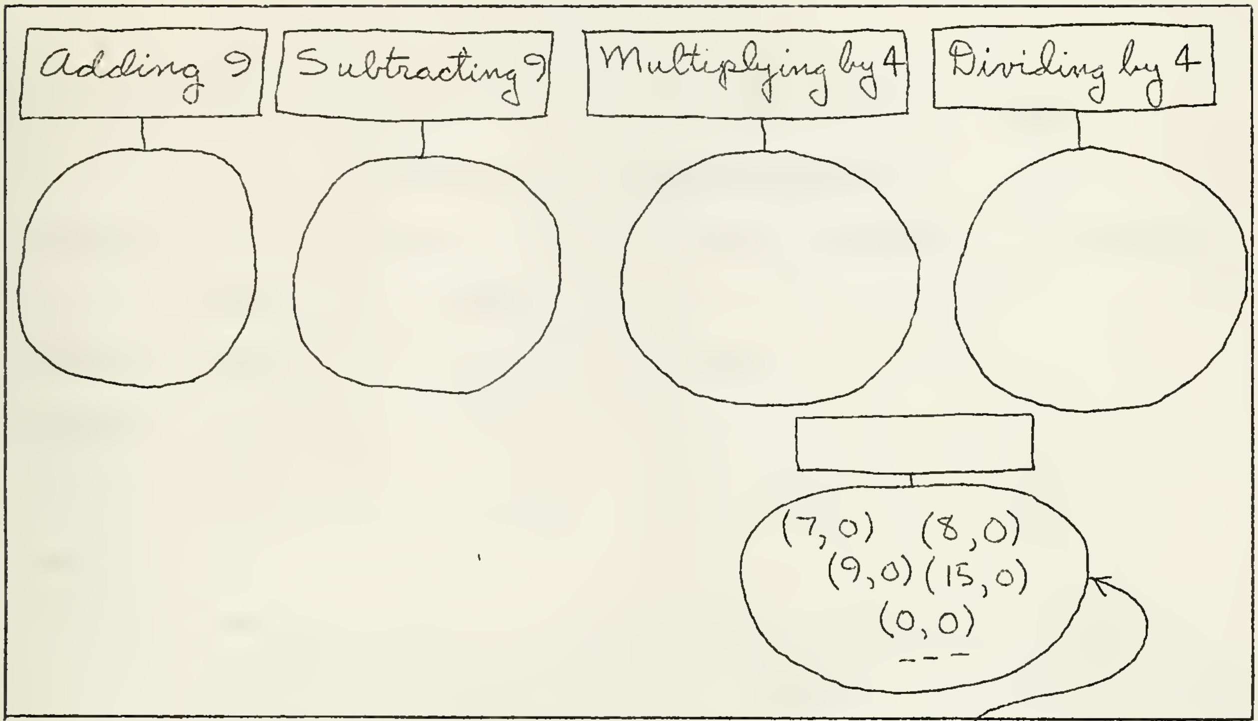
Teacher: Right, you <u>reverse</u> the pairs. That's the way it works. Reverse the pairs. Is there any pair that belongs to both of these last two books?

Student: (0, 0)

Teacher: Is there any pair that belongs to Adding 9 and Multiplying by 4?

Student: (3, 12)

Teacher: Let's look at another book.

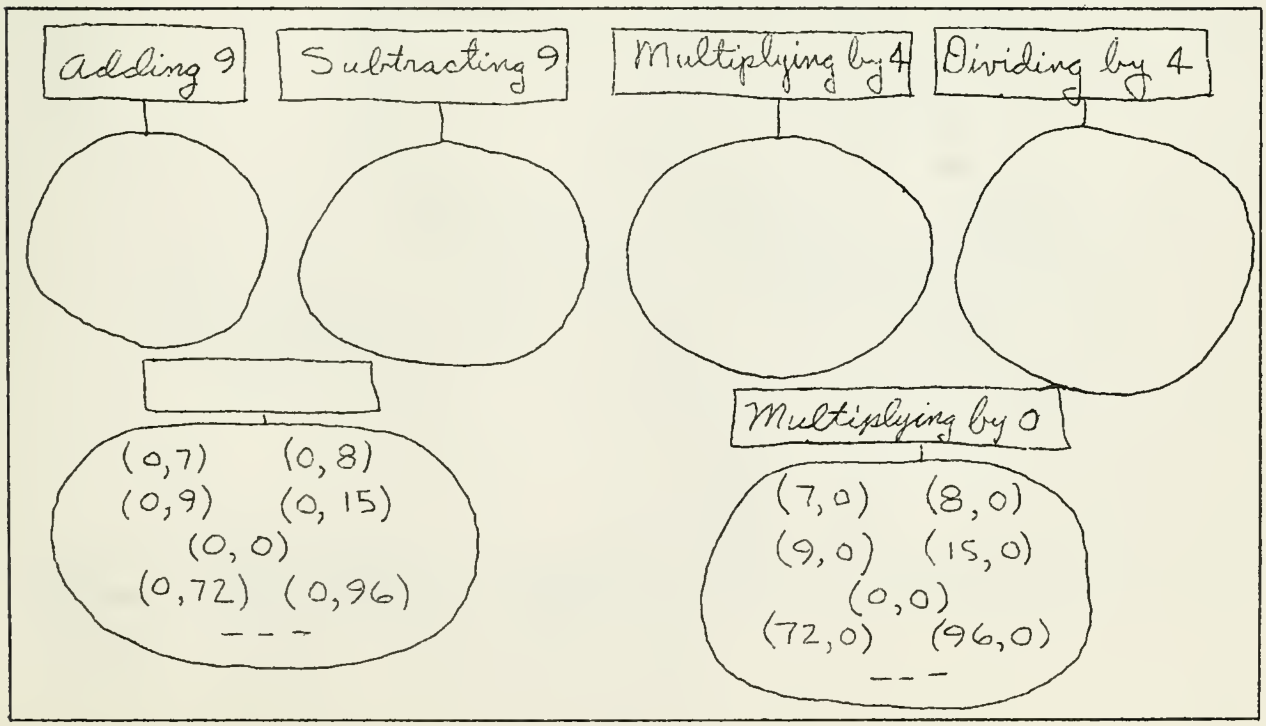


What are some other pairs that belong to this book?...

What is a name for this book?

Student: Multiplying by 0.

Teacher: Let's reverse these pairs. What pairs will we have then?



Teacher:	Which book would you use to do this problem?

74 + 9 = ____

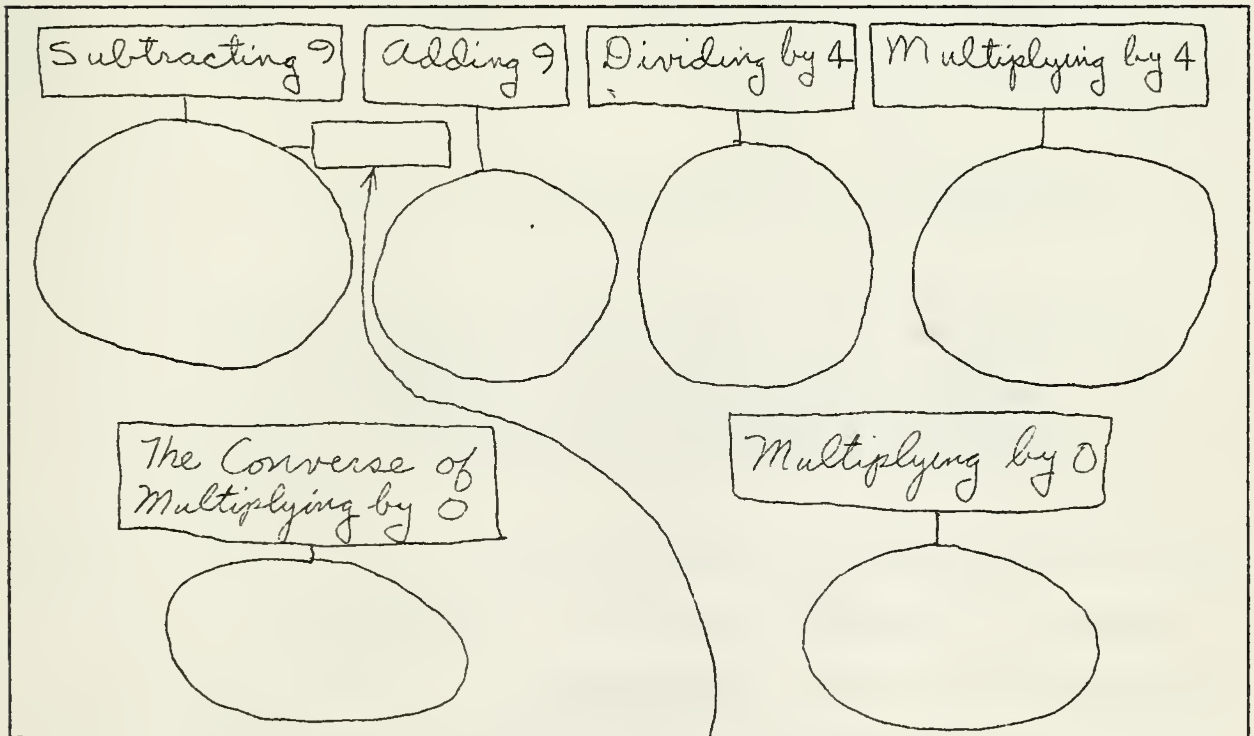
98 - 9 = ____

15 × 0 = ____

What kind of problem could you work using this book?

[Pointing to the unlabeled balloon.]

Student:	There isn't any. You'd never know which pair to pick because they all start with 0.
Teacher:	Right, Then can you divide by 0?
Student:	No.
Teacher:	Can anyone divide by 0?
Student:	No.
Teacher:	However, there is a name for this last book. It's called 'The Converse of Multiplying by 0'. Converse. C-O-N-V-E-R-S-E. Can you give me another name for this book?



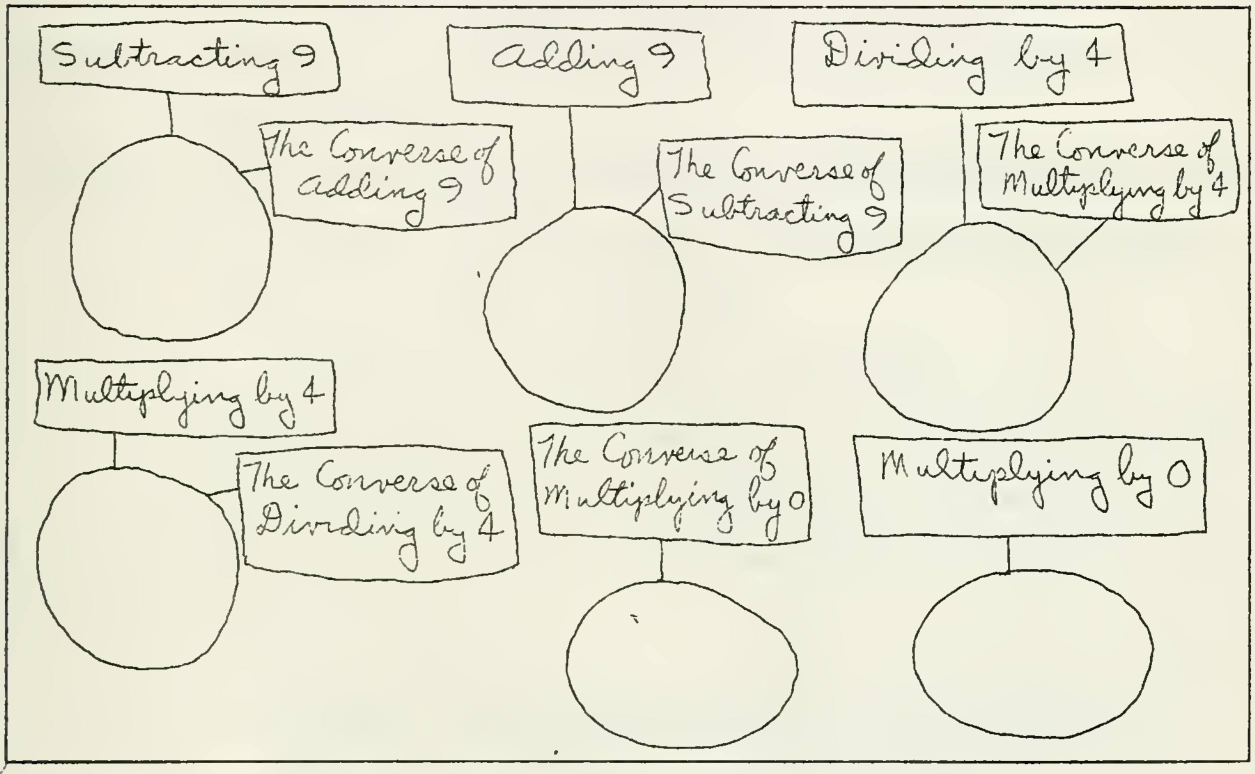
Student:	The Converse of Adding 9.
Teacher:	Correct. Let's put that down.

Student: I think it ought to be called "The Reverse of Adding 9".

Teacher: Maybe that would be a good name, but that's not what mathematicians call it. We reverse the pairs and we get the converse. What is another name for the book Adding 9?

Student: The Converse of Subtracting 9.

Teacher: [Writes this in.] What is another name for the book Dividing by 4? Multiplying by 4?



[Students may want to give another name to Multiplying by 0. They may give 'The Converse of the Converse of Multiplying by 0.' This is acceptable. But, of course, 'The Converse of Dividing by 0' is meaningless and not acceptable, since 'Dividing by 0' has no referent.]

Teacher: Notice that in this book, Subtracting 9, if we start with 15, we go to 9 and nowhere else. In the book, Dividing by 5, if we start with 35, we go to 7 and nowhere else. Is this sort of thing true for the book we call 'The Converse of Multiplying by 0'?

Student: No.

Teacher: Is it true for the book, Adding 9? Multiplying by 4? Multiplying by 0?

Student: Yes. Yes. Yes.

Teacher: Books like this, where you go <u>nowhere</u> <u>else</u>, we call 'operations'. Is Dividing by 5 an operation?

Student: Yes.

Teacher: Which of these books are operations? Which are not operations?

Student: All but one are operations. The Converse of Multiplying by 0 is not an operation.

Teacher: Each of you make up a book which is an operation. Draw a loop on your paper and list five pairs which belong to that operation. Give your operation a simple name.

[Check this work some way at this point.]

Now draw another loop. Reverse the pairs that are listed in your first loop and list these reversed pairs in the second loop. Is this set of pairs an operation? Give this set of ordered pairs a name.

[The word 'ordered' can be used here without special emphasis. If students question this, just point out that you are saying 'ordered' because it makes a difference about which number comes first.]

Teacher: We have a special name if both the book and its converse are operations. We use the name 'Inverse' instead of 'Converse'. Inverse. I-N-V-E-R-S-E. Now think about the book, Adding 9. Think about the book, The Converse of Adding 9. What other name have we already given this second book?

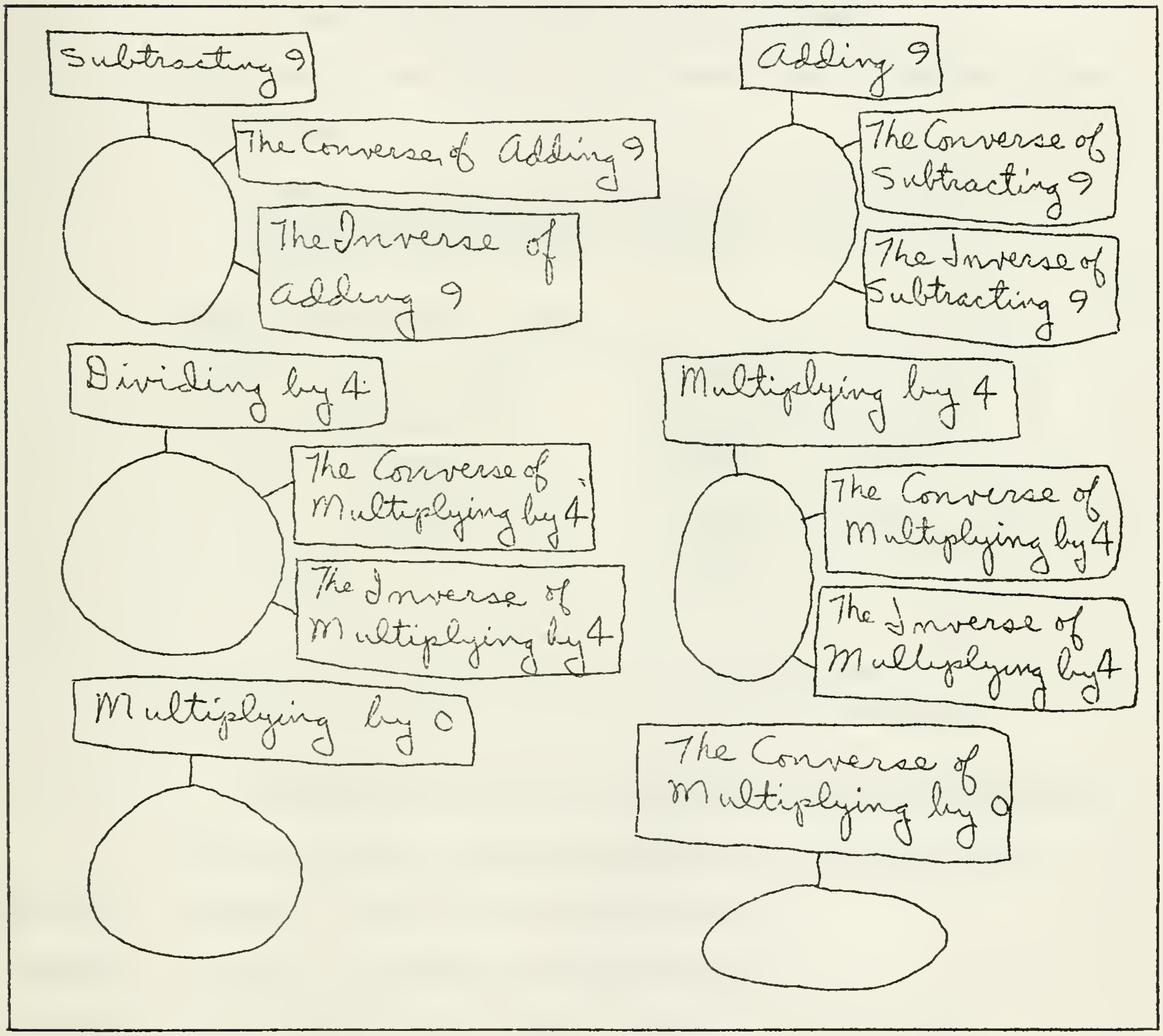
Student: Subtracting 9.

Teacher: What new name can we give this book?

Student: The Inverse of Adding 9. [Write this name in the proper place.]

Teacher: What other name can we give the book Adding 9?

Student:	The Inverse of Subtracting 9. [Write this name in the proper place.]
Teacher:	Is there any one of these books for which we will not use the word 'inverse'?
Student:	Yes. The Converse of Multiplying by 0.
Teacher:	Will we use the word 'inverse' for any other of these books?
Student:	Yes. Dividing by 4 is the Inverse of Multiplying by 4.
Teacher:	Any others?



Teacher:	If the word 'converse' appears in a name, can we always replace the word 'converse' by 'inverse' and get another correct name?

Student: No.

Teacher: If the word 'inverse' appears in a name can we replace the word 'inverse' by 'converse' and get another correct name?

Student: Yes.

Teacher: If we <u>can</u> correctly use the word 'inverse' we usually use it instead of using the word 'converse'. We would seldom say:

Subtracting 9 is the converse of adding 9.

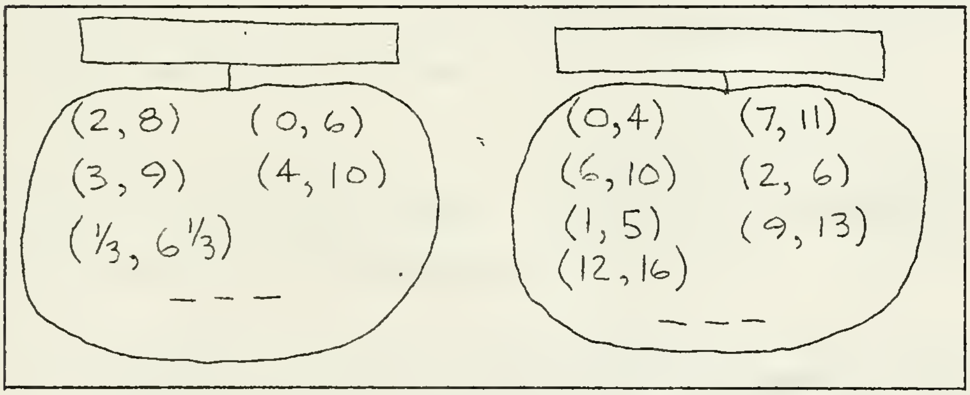
We would usually say:

Subtracting 9 is the inverse of adding 9.

Now look at your papers. Write another name for each of the sets of pairs.

[If the word 'converse' appears and if the word 'inverse' is applicable, ask them to write still another name.]

Here are two sets of pairs.



[Ask them to give more ordered pairs for each operation.]

What is a name for this first operation? For the second?

Student: Adding 6. The Inverse of Subtracting 6.

Student: Adding 4. The Inverse of Subtracting 4.

Teacher: Now. Jane, pick a number. What one did you pick?

Student: 7.

Teacher: Let's go to the first operation. Adding 6.

[Put '(7,)' inside the proper loop.]

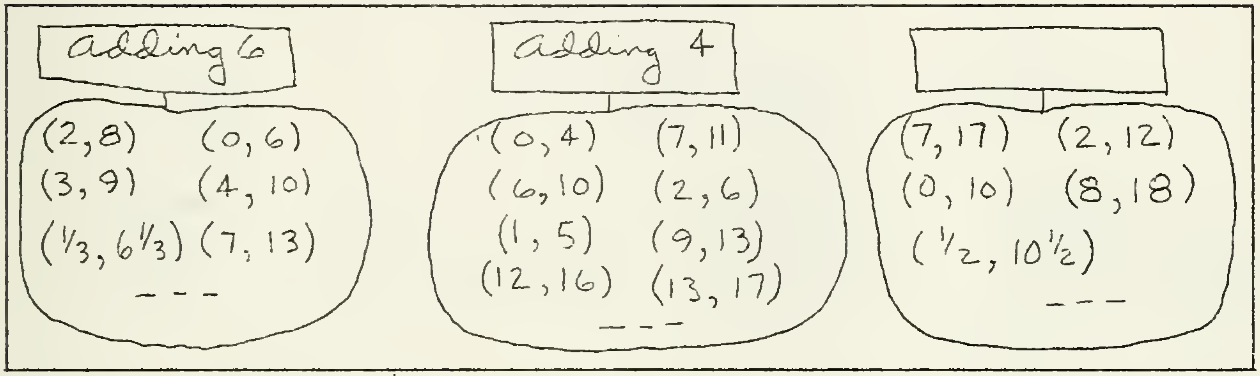
Teacher: Perform this operation on 7. What second number do we get?

Student: 13 [Write '13' in the proper place.]

Teacher: Now perform this second operation, adding 4, on the number 13. [Write '(13,)' inside the proper loop.] What shall I write?

Student: 17

Teacher: So, if I start with 7 [write '(7,)'] and perform the operation, adding 6, then take that result and perform the operation adding 4 on it, I get 17. [Write '17' in the proper place.]



Teacher: Let's pick another number.

Student: 2

Teacher: Perform the first operation. Now perform the second operation on that result. What do you have?

Student: 12

[Now have each child go through this procedure. List the ordered pairs as shown above. Draw a loop around them.]

Teacher: How many such pairs are there? Imagine that we have all such pairs. Do we want to say that this set of pairs is an operation?

Student: Yes.

Teacher: Can you suggest a good name for this operation?

Student: Adding 10.

*

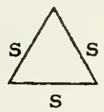
The examples given above have been based on the set of numbers of arithmetic. Examples using real numbers and the operations addition and multiplication on real numbers can also be used.

UICSM-NETRC Math Study Tests

In Newsletters 1 and 2 we presented the first eight tests in the UICSM-NETRC series. Here are the next four, designated by the letters I, J, K, and L. The page of the text to have been completed is again in brackets after the letter. A fifteen minute time limit has been set for each for the purposes of the NETRC Math Study.--R.S.

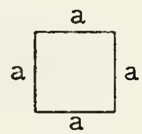
Test I[2-59]

I. Some geometric figures have been drawn and the relative measures of their sides indicated with pronumeral expressions as below. Choose the expression which, when written in the blank in 'P = ____', would yield a correct perimeter formula for the figure.

1. 

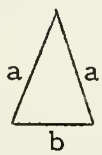
(A) sss (B) 3s

(C) $\frac{s}{3}$ (D) none of these

2. 

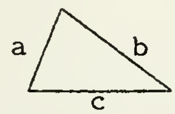
(A) aaaa (B) aa

(C) aa + aa (D) none of these

3. 

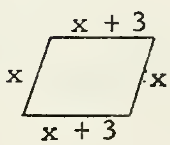
(A) aa + b (B) 2a + b

(C) aab (D) none of these

4. 

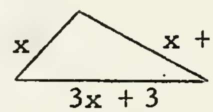
(A) abc (B) 3b

(C) a + b + c (D) none of these

5. 

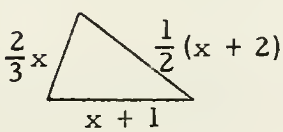
(A) x + 6 (B) xxxx + 6

(C) xx + 6 (D) none of these

6. 

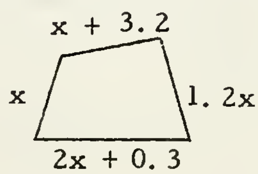
(A) 3x + 10 (B) 3xxx + 10

(C) 5x + 10 (D) none of these

7. 

(A) 2x + 3 (B) $\frac{7}{6}x + 3$

(C) $\frac{13}{6}x + 2$ (D) none of these

8. 

(A) 5.2x + 3.5 (B) 2.4xx + 1.86

(C) 3.2x + 3.5 (D) none of these

II. Choose the equivalent pronumeral expression from among those listed, if there is one.

9. $2a + 3b$

(A) $5ab$ (B) $6ab$ (C) $5(a + b)$ (D) none of these

10. $x(3xy)(5x)$

(A) $9xy$ (B) $15xy$ (C) $15xxxy$ (D) none of these

11. $3(3 + y) + 7(y + 3) + 3 + y$

(A) $11(y + 3)$ (B) $9y + 15$ (C) $10(yyy + 3)$ (D) none of these

12. $(\frac{1}{3}x)(\frac{1}{2}y) + (\frac{1}{2}x)(\frac{1}{3}y)$

(A) $\frac{1}{3}xy$ (B) $\frac{1}{6}xy$ (C) $\frac{5}{3}xxyy$ (D) none of these

13. $y(y + 3) + 3(y + 3)$

(A) $3y(y + 3)$ (B) $(y + 3)(y + 3)$ (C) $yy + 3y + 6$ (D) none of these

14. $xx + 3x + 2$

(A) $x(x + 5)$ (B) $x(x + 3x) + 2$ (C) $(x + 1)(x + 1)$ (D) none of these

15. $a(2a + 3b) + b(3a + 2b)$

(A) $(a + b)(2a + 3b)$ (B) $2(aa + 3ab + bb)$

(D) $(a + b)(3a + 2b)$ (D) none of these

16. $xx + 2xy + yy$

(A) $(x + y)(x + y)$ (B) $x[(x + 2y) + y]$

(C) $x(x + 2)y(2x + y)$ (D) none of these

III. Choose the correct completion for a true generalization.

17. For each x, the sum of $(3x + 2)$ and the product of 4 by $(5x + 3)$ is

(A) $17x + 11$ (B) $23x + 14$ (C) $60xx + 6$ (D) none of these

18. For each x, for each y, the product of the sum of $(x + y)$ and y by the sum of $(2x + y)$ and $(y + x)$ is

(A) $(2x + 2y)(x + y)$ (B) $3xx + 4yy$

(C) $(x + 2y)(3x + 2y)$ (D) none of these

IV. Choose the correct reason for the numbered step.

$$\left.\begin{aligned}&(2x + 3y)(x + y)\\ &= (2x + 3y)x + (2x + 3y)y\end{aligned}\right\}\ 19.$$

(A) apm (B) dpma (C) ℓdpma (D) none of these

$$\left.\begin{aligned}&= 2xx + 3yx + (2xy + 3yy)\\ &= 2xx + 3yx + 2xy + 3yy\end{aligned}\right\}\ 20.$$

(A) cpm (B) apm (C) cpa (D) none of these

Key for Test I[2-59]:

1. B	2. D	3. B	4. C	5. D	6. C	7. C
8. A	9. D	10. C	11. A	12. A	13. B	14. D
15. B	16. A	17. B	18. C	19. C	20. D	

Test J[2-80]

I. Which of the given expressions, if any, will yield a true generalization when written in the blank?

1. $\forall_x\ -(3 - x) =$ _____.

   (A) $-3 - x$ (B) $x - -3$ (C) $x - 3$ (D) none of these

2. $\forall_x \forall_y\ 13x - 5y - 7x + 3y =$ _____.

   (A) $8xy - 4xy$ (B) $6x - 2y$ (C) $6xx - 2yy$ (D) none of these

3. $\forall_x \forall_y\ (3x - 4y) - (y - 2x) =$ _____.

   (A) $5(x - y)$ (B) $x - 5y$ (C) $5x + 3y$ (D) none of these

4. $\forall_x \forall_y\ x - y(x - y) =$ _____.

   (A) $x - (x + y)y$ (B) $x - xy - yy$ (C) $x - xy + yy$ (D) none of these

5. $\forall_x \forall_y \forall_z\ x - (y - z + x) =$ _____.

   (A) $2x - y - z$ (B) $-(y + z)$ (C) $z - y$ (D) none of these

6. $\forall_x \forall_y\ 3(2x - 3y) - 5(x - 2y) =$ _____.

   (A) $x - y$ (B) $x + y$ (C) $x - 19y$ (D) none of these

7. $\forall_x \forall_y\ (x + y)(x - y) =$ _____.

   (A) $xx - xy + yy$ (B) $x(x - y) + y(x - y)$

   (C) $xx + xy - yy$ (D) none of these

8. $\forall_x \forall_y \forall_z \ (x - y)(x - z) =$ _____ .

(A) $xx + yz$ (B) $(y - x)(z - x)$ (C) $xx - yz$ (D) none of these

9. $\forall_x \forall_y \ x(x + -y) + -y(x + -y) =$ _____ .

(A) $(x - y)(x - y)$ (B) $xx - 2xy - yy$

(C) $xx - xy - yy$ (D) none of these

10. $\forall_x \ (x - 1)(x - 2) =$ _____ .

(A) $xx + 2$ (B) $xx - 3$ (C) $xx + 3x - 2$ (D) none of these

II. What is the correct reason for each step in this proof of:

$$\forall_x \forall_y \ -(xy) = -xy$$

$xy + -xy$

. . . .

11. (A) ps
(B) po
(C) ℓdpma
(D) none of these

$= (x + -x)y$

. . . .

12. (A) $\forall_x \forall_y \ -xy = (-x)y$
(B) po
(C) ps
(D) none of these

$= 0y$

. . . .

13. (A) pa0
(B) cpm
(C) pm0
(D) none of these

$= y0$

. . . .

14. (A) $\forall_x \ 0x = 0$
(B) $\forall_y \ y + 0 = y$
(C) $\forall_z \ 0 = z0$
(D) none of these

$= 0$

15. (A) po
(B) $\forall_x \forall_y \ -xy = x(-y)$
(C) 0-sum theorem
(D) none of these

Hence, $xy + -xy = 0$

So, $-(xy) = -xy$

III. What is the correct reason for each step in this proof of:

$$\forall_x \forall_y \forall_z\ x(y - z) = xy - xz$$

$x(y - z)$

16. (A) po
    (B) ps
    (C) $\forall_x \forall_y\ x - y = -(y - x)$
    (D none of these

$= x(y + -z)$

17. (A) dpma
    (B) ℓdpma
    (C) $\forall_x \forall_y \forall_z\ x(y - z) = xy - xz$
    (D) none of these

$= xy + x(-z)$

18. (A) $\forall_x \forall_y\ x(-y) = -xy$
    (B) $\forall_x \forall_y\ x(-y) = -(xy)$
    (C) $\forall_x \forall_y\ -xy = -(xy)$
    (D) none of these

$= xy + -xz$

19. (A) $\forall_x \forall_y\ -xy = x(-y)$
    (B) $\forall_x \forall_y\ -xy = -(xy)$
    (C) $\forall_x \forall_y\ -(xy) = x(-y)$
    (D) none of these

$= xy + -(xz)$

20. (A) $\forall_x \forall_y\ x - y = x + -y$
    (B) po
    (C) $\forall_x \forall_y\ -xy = -(xy)$
    (D) none of these

$= xy - xz$

Key for Test J[2-80]:

1. C	2. B	3. A	4. C	5. C	6. B	7. B
8. B	9. A	10. D	11. D	12. B	13. B	14. C
15. C	16. B	17. B	18. A	19. B	20. A	

Test K[2-85]

1. Which of these operations does <u>not</u> have an inverse?

   (A) sameing (B) oppositing

   (C) absolute valuing (D) adding 0

2. Which of these operations does <u>not</u> have an inverse?

   (A) multiplying by 0 (B) adding 1

   (C) oppositing (D) multiplying by 1

3. Which kind of problem always has an answer?

   (A) division of real numbers (B) division of numbers of arithmetic

   (C) subtraction of real numbers (D) subtraction of numbers of arithmetic

4. These pairs belong to certain operations. Which operation does <u>not</u> have an inverse?

   (A) (2, 4), (−2, 4) (B) (2, 5), (5, 8)

   (C) (0, −5), (5, 0) (D) (7, 7), (−3, −3)

5. Which of these generalizations is <u>false</u>?

   (A) $\forall_x\ x = x$ (B) $\forall_x \forall_y$ if $x = y$ then $y = x$

   (C) $\forall_x \forall_y \forall_z$ if $x = y$ and $y = z$ then $x = z$ (D) none of these

6. Which generalization states that subtracting is the inverse of adding?

   (A) $\forall_x \forall_y\ (x + y) - y = x$ (B) $\forall_x \forall_y\ (x - y) + y = x$

   (C) $\forall_x \forall_y\ x + -y = x - y$ (D) none of these

7. Which of these expressions is <u>not</u> a numeral?

   (A) $\frac{0}{0 + 3}$ (B) $\frac{3 + 3}{3 - 3}$ (C) $\frac{3 \times 0}{3 \times 3}$ (D) $\frac{0 + 3}{0 - 3}$

8. Which of these generalizations was <u>not</u> a basic principle?

   (A) pa0 (B) pm0 (C) pm1 (D) po

9. A new operation, indicated by a '☆', is to be carried out in one of the ways shown below. In which case will it be commutative?

   (A) $x ☆ y = x + y + xy$ (B) $x ☆ y = x + y + y$

   (C) $x ☆ y = x + yy$ (D) $x ☆ y = xxy$

10. A new operation, indicated by a '$*$', is to be carried out in one of the ways shown below. In which case will it be commutative?

(A) $x * y = (xx + yy) \div (x + y)$ (B) $x * y = xx \div yy$

(C) $x * y = xx - yy$ (D) $x * y = (x - y) - (y - x)$

11. For each real number x there is a real number x^* such that $x + x^* = 0$. Then, it is not the case that, for each x,

(A) $x - x^* = 2x$ (B) xx^* is positive

(C) $xx \geq xx^*$ (D) x^*x^* is nonnegative

12. For each nonzero real number x there is a real number $\overline{x}$ such that $x\overline{x} = 1$. Then, for each nonzero x, $\overline{x}$ is the same as

(A) 1 (B) $1 \div x$ (C) $-x$ (D) 0

13. Which of the following is a true generalization?

(A) distributive principle for addition over subtraction

(B) distributive principle for subtraction over addition

(C) distributive principle for subtraction over subtraction

(D) none of them

14. Which of the following is not a true generalization?

(A) distributive principle for absolute valuing over addition

(B) distributive principle for oppositing over addition

(C) distributive principle for oppositing over subtraction

(D) distributive principle for sameing over subtraction

15. For each x, for each y, __?__ + x = y.

(A) $x - y$ (B) $-(x + y)$ (C) $y - x$ (D) none of these

16. For each x, for each y, x - __?__ = y.

(A) $x - y$ (B) $y - x$ (C) $-(x + y)$ (D) none of these

17. Which "cancellation principle" is not a true generalization?

(A) $\forall_x \forall_y$ if $-x = -y$ then $x = y$

(B) $\forall_x \forall_y$ if $|x| = |y|$ then $x = y$

(C) $\forall_x \forall_y \forall_z$ if $x + z = y + z$ then $x = y$

(D) $\forall_x \forall_y \forall_z$ if $x - z = y - z$ then $x = y$

18. Which "cancellation principle" is <u>not</u> a true generalization?

(A) $\forall_x \forall_y$ if $13x = 13y$ then $x = y$

(B) $\forall_{x \neq 0} \forall_{y \neq 0}$ if $0 \div x = 0 \div y$ then $x = y$

(C) $\forall_x \forall_y \forall_z$ if $x - z = y - z$ then $x = y$

(D) $\forall_x \forall_y \forall_{z \neq 0}$ if $xz = yz$ then $x = y$

19. Suppose there is an operation called 'bracketing', such that, for each y, bracketing with y has an inverse. What else can you conclude about bracketing?

(A) it is commutative (B) it is associative

(C) it has a cancellation principle (D) nothing

Key for Test K[2-85]

1. C	2. A	3. C	4. A	5. D	6. A	7. B
8. B	9. A	10. A	11. B	12. B	13. D	14. A
15. C	16. A	17. B	18. B	19. C		

Test L[2-108]

I. Choose the correct simplification, if it is given

1. $\frac{3a}{-4b} \circ \frac{2bb}{-3}$

(A) $\frac{ab}{2}$ (B) $\frac{a}{2}$ (C) $\frac{ab}{-2}$ (D) none of these

2. $\frac{x}{3} - \frac{x}{2}$

(A) $\frac{x}{1}$ (B) $\frac{1}{6}$ (C) $\frac{x}{6}$ (D) none of these

3. $\frac{a}{5} + \frac{a}{3}$

(A) $\frac{a}{4}$ (B) $\frac{a}{8}$ (C) $\frac{aa}{8}$ (D) none of these

4. $\frac{12uu}{2vv} \div \frac{3u}{v}$

(A) $\frac{2u}{v}$ (B) $\frac{18uuu}{vvv}$ (C) 2 (D) none of these

5. $\frac{7}{2rs} + \frac{3}{8rst}$

(A) $\frac{7t + 3}{rst}$ (B) $\frac{14t + 3}{4rst}$ (C) $\frac{28t + 3}{8rst}$ (D) none of these

6. $(\frac{2}{5} + \frac{3}{4}) \div (\frac{1}{10} + \frac{1}{4})$

(A) $\frac{3}{2}$ (B) $\frac{161}{400}$ (C) 12 (D) none of these

7. $1 - \frac{x - 5}{2x - 3}$

(A) $\frac{x - 8}{2x - 3}$ (B) $\frac{x + 2}{3 - 2x}$ (C) $\frac{x + 2}{2x - 3}$ (D) none of these

8. $\frac{2}{y - 7} - \frac{3}{y - 5}$

(A) $\frac{-y - 12}{(y - 7)(y - 5)}$ (B) $\frac{11 - y}{(7 - y)(5 - y)}$ (C) $\frac{-y - 31}{(y - 7)(y - 5)}$ (D) none of these

II. Choose the correct answer, if it is given.

9. Which of the following is not a true principle?

(A) distributive principle for multiplication over subtraction

(B) left distributive principle for multiplication over subtraction

(C) distributive principle for division over subtraction

(D) left distributive principle for division over subtraction

10. Which of these numbers is a counter-example to the generalization:

$$\forall_{x < 0} \; x + \frac{1}{x} \leq -2$$

(A) -1 (B) 0 (C) 1 (D) none of them

11. Suppose 'x' is given larger and larger positive values. Then what happens to the values of '$\frac{4}{x + 3}$'?

(A) they increase (B) they decrease
(C) they stay the same (D) cannot tell

12. Suppose 'x' is given larger and larger positive values. Then what happens to the values of '$\frac{2}{3 + \frac{6}{x}}$'?

(A) they increase (B) they decrease
(C) they stay the same (D) cannot tell

13. Which pair of expressions are equivalent?

(A) $\frac{2x}{y} + \frac{3y}{x}$; $\frac{2x + 3y}{x + y}$ (B) $\frac{x}{y}$; $\frac{x + 1}{y + 1}$

(C) $\frac{1}{3x - y}$; $-\frac{1}{y + 3x}$ (D) none of them

14. $\forall_{a \neq 0} \forall_b \; a \cdot$ ______ $= b$

(A) $\frac{b}{a}$ (B) $\frac{a}{b}$ (C) $\frac{1}{a}$ (D) none of these

15. $\forall_{x \neq 0} \forall_{y \neq 0} \; x \div$ ______ $= y$

(A) y (B) $\frac{x}{y}$ (C) $\frac{y}{x}$ (D) none of these

16. $\forall_{a \neq 0} \forall_{b \neq 0} \; (a \div b) \div$ ______ $= b \div a$

(A) $\frac{b}{a}$ (B) $\frac{bb}{aa}$ (C) $\frac{aa}{bb}$ (D) none of these

17. For each nonzero real number x there is a real number $\overline{x}$ such that $x\overline{x} = 1$. Then, it is <u>not</u> the case that, for each nonzero x,

(A) $x \div \overline{x} = xx$ (B) $\overline{x} \div x = x \div \overline{x}$

(C) $x + \overline{x} = (xx + 1)\overline{x}$ (D) $\overline{x\overline{x}} = 1$

18. Which generalization tells you that, given a first number [other than 0] and a second number, there is a number whose product by the first is the second?

(A) the zero-product theorem (B) the principle of quotients

(C) the division theorem (D) none of these

Key for Test L[2-108]

1. A	2. D	3. D	4. A	5. C	6. D	7. C
8. B	9. D	10. D	11. B	12. A	13. D	14. A
15. B	16. C	17. B	18. B			

THE NEW UNITS 7 AND 8

The latest revisions of the UICSM units titled 'Mathematical Induction' and 'Sequences' are now in active preparation and are scheduled for publication by the University of Illinois Press in the summer of 1961. Unit 7 is intended to provide the textual basis for about the first half of the third year of study in UICSM mathematics, while Unit 8 contains material for the remainder of the year.

The titles may be somewhat misleading in that they do not suggest the full range of mathematics encompassed by the units. The purpose of this article is to survey the structure and content of the two units, noting the arrangement and major topics of each.

Briefly, Unit 7 is a revision and expansion of the material in the old unit on mathematical induction preceding the introduction of sigma-notation. Unit 8 begins with sigma-notation and includes the material of the old Exponents and Logarithms unit up to the introduction of rational exponents. [The remaining material in the latter unit will be covered in Unit 9, whose publication is scheduled for 1962.]

Unit 7, Mathematical Induction, of course presupposes Units 1 through 6--in particular, an understanding of the material covered in the appendix on logic in Unit 6 (published, along with Unit 5, by the University of Illinois Press last summer).

The unit opens by posing some attention-getting problems on progressions and series, then presents a short section on computational procedures in place-value numeration systems and algebraic algorithms for muliplication and division of polynomials, designed as the review-with-extensions probably needed by most students at this point.

Then the text launches into the main business of the unit: a detailed study of the beginnings of the "finer structure" of the real number system, and especially of the set of integers. To this end, the student is first led to a consideration of the inadequacy of the ten basic principles for real numbers developed in Unit 2 for distinguishing between positive and negative numbers. Four basic principles for positive numbers are then formulated and these are connected to the '>' relation by another assumption. Work on theorems and transformation principles for inequations follows.

The postulates necessary for investigation of the set of positive integers are then motivated and three more basic principles introduced, the third being an induction principle that leads directly to proofs by mathematical induction. Proofs of theorems about positive integers are then carried out on the basis of recursive definitions, and applications are made to polygonal numbers and combinatorial problems.

The existence of a least member in every nonempty set of positive integers is then proved and the existence of an integer greater than any given real number is postulated. The postulate, labeled the 'Cofinality Principle', corresponds roughly to that traditionally known as the axiom of Archimedes (or Eudoxus). It is needed principally for the introduction of the greatest integer function.

Divisibility is defined in another principle and the Highest Common Factor algorithm developed, then applied to solving some Diophantine problems.

Many lists of Miscellaneous Exercises of mixed types and varying degrees of difficulty are interspersed throughout the unit to provide opportunity for application of old and new concepts. These exercises also serve to maintain skills in manipulation and techniques of problem-solving. A variety of geometrical and worded problems are included. There is also an extensive list of Review Exercises near the end of the unit.

Finally, there is appended, for easy reference, a list of basic principles and theorems mentioned in the unit. The list contains the ten basic principles of Unit 2, the additional assumption that $1 \neq 0$, the 78 theorems of Unit 2 as given on pages TC[2-61]c through i, and the 51 additional theorems proved in Unit 7. The new principles adopted in this unit are also given at their appropriate places.

Unit 8, Sequences, opens with a section on sigma-notation, as mentioned above, then takes up summation of sequences, both directly and by the constant difference method. Arithmetic progressions and means are considered, then the Pigeonhole Principle.

The multiplicative part follows, with an introduction to the pi-notation for products, more work on integral exponents, and developments of both finite and infinite geometric progressions. Finally, there are sections on factorials, further combinatorial problems, and the binomial theorem for integral exponents. --R.S.

* * *

The object of a mathematical education is to acquire the powers of analysis, of generalization, and of reasoning.

To teach mathematics is to teach logical precision. A mathematical teacher who has not taught that, has taught nothing.

The habit of logical precision is the instinct for the subtle difficulty.

--Alfred North Whitehead
'The Principles of Mathematics in Relation to Elementary Teaching' (1913)

*

CORRECTION: We regret that several errors in the use of punctuation and quotation marks remained in our last issue despite what was (we thought) a conscientious job of checking and proofreading. Most of them were in the article titled '$\sqrt{8}$: Rational or Irrational?', including this major erratum: the fourth line from the bottom on page 18 should have read 'integer and 2p is an integer. So, (8q - 2p) is an integer.' We thank the alert and helpful readers who pointed this out. Corrections from readers are always gratefully received, of course, though in our immediate chagrin we may feel a bit like awarding the prize of a one-way ticket to Zabranchburg to the one submitting the longest list. --Ed.

NEWS AND NOTICES

We are again pleased to note here the professional activities of UICSM teachers that have come to our attention lately through their weekly reports and letters to the project office. We hope that all UICSM teachers will keep us posted on their contributions to mathematics education, whether large or small.

Mrs. Mary S. Huzzard, Cheltenham High School, Wyncote, Pa., spoke to the junior and senior high school mathematics and science teachers of Jenkintown, Pa., on the UICSM courses; also to the fourth, fifth, and sixth grade teachers there.

Mrs. Ruth Wong, University of Hawaii High School, Honolulu, spoke on UICSM in connection with a seminar of practice teachers and cooperating teachers in the Honolulu public schools. The seminar included a symposium on recent curriculum trends. Mrs. Wong also presented UICSM to a General Semantics group there, which, she reports, "was highly interested in and enthusiastic about emphasis on precision in language and nonverbal awareness." An interesting "public relations" sidelight: she further notes that "There was also one individual who seemed intent on having me make sweeping claims for the program only to 'prove' I was wrong. I avoided most of these leading questions by refusing to give a pat 'yes' or 'no' as he seemed to want me to do."

Mr. Eugene Epperson, Talawanda High School, Oxford, Ohio, conducted a workshop for junior and senior high school teachers at Wilmington, Ohio, on February 28. He discussed the introduction of basic principles and the solution of equations and worded problems as per Units 1 and 3. The workshop was sponsored by the Ohio Council of Teachers of Mathematics Workshop Consultant Service.

Mr. William Annett of Seaford Junior-Senior High School, Seaford, Long Island, New York, reports he is conducting an adult education course in Units 1-4. Fifteen parents are enrolled in the course, most of whom have children taking First Course at Seaford. The course was requested by parents and begun during the second semester in order to "...insure no parental interference with discovery on the part of the children." Mr. Annett plans to devote ten meetings (of two hours each) to Units 1 and 2 this semester and continue with Units 3 and 4 during the first semester of 1961-62, then begin again with a new group. "These parents exhibit <u>almost</u> as much interest as the children," he adds.

Sister Mary Rosaria, St. Basil High School, Pittsburgh, presided at the mathematics section meeting of the diocesan secondary school convention on February 3.

Sister Grace Marie of the Villa Maria Academy, Erie, Pa., was in Pittsburgh during the week ending February 17 as a member of the evaluation committee visiting Elizabeth Seton High School.

Mr. Richard Fleischer, Johnson Regional High School, Clark, New Jersey, taught his Unit 6 (10th-grade) class as a demonstration class at New Jersey State Teachers College on March 11.

Mr. Fred H. Green of the North Plainfield (New Jersey) High School has been appointed Assistant Director of the NSF-sponsored Summer Mathematics Institute at Baldwin-Wallace College, Berea, Ohio, this summer. He will conduct the seminar on "Curriculum Reform Proposals and Their Implementation" in which UICSM, SMSG, and Ball State materials will be examined and in which he hopes to "promote plenty of discussion and discovery." With respect to UICSM, Mr. Green observes that "I personally meet too many uninformed experts on our program... those who know very little about what we are doing, yet condemn isolated segments of the material."

*

Mr. Beberman traveled extensively during February and March, visiting UICSM and NETRC (film project) teachers, delivering speeches and lectures, and conducting demonstration classes in a number of communities across the nation. His February schedule included three engagements in Florida, an SMSG conference in Chicago, and visits to four locations in the state of Washington. After several more engagements in Washington and Oregon during the early part of March he spent about a week in Alaska, making presentations to several teacher's groups there, then squeezed in several speeches and demonstrations in California before returning to do the same at a Rockford, Illinois, teacher's institute last week. He has just returned from another week-long trip and will be away on yet another next week.

Miss Hendrix also traveled extensively in the past two months in connection with her research on nonverbal awareness and the psychology of learning. She showed some of the NETRC films to several university audiences in Utah during February, delivering an address or conducting a discussion each time on the significance of the film for the psychology of learning. Similar programs were also conducted in Oregon, California, and Indiana during February. She was interviewed in connection with her research interests over the University of Illinois television station on March 14. Her plans include a presentation at the annual meeting of the National Council in Chicago on April 7, attendance at a Washington, D. C., conference on productive thinking in education April 27-29, and several engagements in New York, Massachusetts, and Connecticut in May--including a research consultation with Margaret Mead and the staff of Seeing-Eye, Inc., in New Jersey.

Mr. Arnold Petersen, Teacher Associate with the project this year while on leave from his position as Head of the Mathematics Department at the Pascack Valley Regional High School, Hillsdale, New Jersey, visited the Lakeview Public Schools of St. Clair Shores, Michigan as the project representative on February 23 and 24. He later visited his own school to discuss expansion of the UICSM program, then began work on an attempt to organize an NSF summer institute in northern New Jersey or the New York metropolitan area for 1962. He visited Talawanda High School, Oxford, Ohio, on March 3 en route to Urbana.

Miss Eleanor McCoy visited the laboratory school at Western Illinois University on March 13, then spent the next two days at Pekin High School, where she had taught before joining the UICSM staff. She also visited five UICSM schools in northern Illinois during the week of March 20.

UICSM Newsletter

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UNIVERSITY OF ILLINOIS COMMITTEE ON SCHOOL MATHEMATICS

1208 West Springfield

Urbana, Illinois

UICSM Newsletter No. 5

May 19, 1961

A NOTE TO UICSM TEACHERS

This is the fifth and last Newsletter to be published by UICSM during the 1960-61 school year. The Newsletter was conceived principally as an avenue of communication from the project office to you, the teachers using UICSM text materials. We have tried to keep it as practical and useful as possible, thinking this might best be done by publishing materials that would supplement and amplify the Teacher's Commentaries.

Your response has been gratifying and encouraging. Our hope and intention is to continue to meet your needs in any way possible. You can help us do this by continuing and increasing the "feedback" to the UICSM office in the form of letters containing comments, pedagogical suggestions, expository articles, test items, and accounts of experiences in teaching and administering the UICSM program.

Newsletter No. 6, to appear early in September, is scheduled to contain a description of what we would suggest as a minimum course of study in each of the first six units if there is not time for complete coverage. The units seem to expand with each revision, but the school year does not; several of you have felt the squeeze and asked for advice on which sections are essential and which less essential.

We look forward to meeting those of you who are coming to the NSF Summer Institute in Urbana for the first time next month, and to renewing friendships with those who have been here before. If you are traveling this summer and have an opportunity to visit us in Urbana, we will, of course, be very pleased to meet you, to show you how we operate, and to discuss with you any phase of the UICSM program. In any case, have a pleasant and rewarding summer.

TEACHING FIRST COURSE

by Sister Mary Sarah, S. S. N. D.*

After attendance at in-service mathematics sessions in Milwaukee, summer institutes, mathematics lectures, and a workshop at Notre Dame of the Lake Training Center last August, the impetus came to adopt some project to improve Messmer High School's mathematics program. One of the teachers attended a demonstration course in the teaching of the UICSM First Course at an NSF Summer Institute at Sacramento and felt that this project could be successfully carried out.

Our Principal agreed that two teachers (each with a heterogeneous group) might use the course set up by the University of Illinois Committee on School Mathematics. The first problem seemed to be to answer the student's worry as to why they were singled out. "Is it because I am a poor student?" or "a bright student?" It was neither reason. Each of the two teachers had been assigned a single algebra class and was eager to use the course with this one group. Once such questions were satisfactorily answered there was no further obstacle.

Students seem to have less difficulty with the UICSM course than with traditional algebra. Getting used to the terminology and generalizations, etc., poses more of a problem for the teacher, grounded in traditional mathematics, than it does for the students. But the Teacher's Commentary accompanying the text adequately provides:

1) background information
2) a wonderful help in solving the exercises
3) keys for the exercises
4) supplementary and miscellaneous exercises
5) tests which help determine the strengths and weaknesses of students in the areas taught

Algebra taught in this way seems to have much more meaning for the student. Opportunity for discovery is constant throughout the course. Exercises are well-graded, from the easy to the more difficult. Principles are introduced early and gradually, constantly providing a basis for work. They seem to tie together the ideas and to help students see continuity in their study.

Interest is sustained throughout through a novel use of illustrations. For example: 1) the use of pronumerals, beginning with frames to generate other expressions; 2) the use of a diagram of a camera and a projector in teaching the multiplication of real numbers; 3) the use of 'distance' and 'direction' in taking trips to make adding positive and negative numbers more

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* Sister Mary Sarah teaches at Messmer High School in Milwaukee, Wisconsin. She will attend the NSF Summer Institute for UICSM teachers in Urbana this summer. This article, republished here with her permission, originally appeared in the February, 1961, Wisconsin Teacher of Mathematics.

understandable; 4) using the raised ' - ' sign to distinguish between the operation of adding a negative number and the subtraction operation.

Traditional difficulties are obviated by the manner of teaching the use of grouping symbols and of preparing for teaching exponents. Even lengthy and seemingly involved expressions cease to look formidable. Some student will generally attempt unravelling the expression by applying various principles or the convention. The old exponent mystery has disappeared.

With regard to initiating the course, once the UICSM was apprised of our plan to teach First Course we were given every assistance. The Newsletters, periodically sent out, provide teacher's experiences, excellent suggestions, and tests for the teacher's use.

We have found teaching the course a satisfying and valuable experience for both students and teachers. Of course, since it is our first experience, our evaluation is based on the subject matter taught thus far--the first one and one-half units of the four required.

* * *

ON ORDERING TEXT MATERIALS

Schools intending to order UICSM text materials for use during the 1961-62 school year may wish to take note of this up-to-date summary of their availability.

Unit	Descriptive title
1	The arithmetic of the real numbers
2	Generalizations and algebraic manipulation
3	Equations and inequations
4	Ordered pairs and graphs
5	Relations and functions
6	Geometry
7	Mathematical induction
8	Sequences
9	Exponential and logarithmic functions

The Student's Editions of Units 1, 2, 3, and 4 may be purchased separately at $1.00 per unit after July 1. The four-unit set and the Teacher's Edition of this set will thereafter be sold at $3.00 and $6.00 per copy, respectively.

Units 5 and 6 have also been published at the following prices per copy:

	Student's Edition	Teacher's Edition
Unit 5	1.50	3.00
Unit 6	2.00	4.00

The new Units 7 and 8 were described in Newsletter No. 4, pages 30-31, and will become available from the Press this summer. The estimated per-copy prices (subject to change) of these units are:

	Student's Edition	Teacher's Edition
Unit 7	1.25	2.75
Unit 8	1.75	3.75

Unit 9 is scheduled to appear in January of 1962; schools planning to use it can expect delivery near the beginning of the second semester. The unit begins with rational exponents, then moves into a development of the themes suggested by the title.

All the units mentioned above will be available from

The University of Illinois Press
Urbana, Illinois

and should be ordered by unit number rather than course number or title. (There is a 10% discount on student editions when ordered in lots of 50 or more copies.)

The units on Circular Functions and Complex Numbers will be available without charge from the UICSM Project office. Requests for copies should be sent to UICSM, 1208 West Springfield, Urbana, Illinois. Schools needing the "old" unit on Exponents and Logarithms before the new Unit 9 is ready should write to Max Beberman to make special arrangements. --R.S.

* * *

As the word "obvious," so also the word "proof" has a meaning which is dependent on the audience for whom the proof is intended. All that is required of a proof is that it convince the audience of the truth of the implication at hand.

--R. B. Kershner & L. R. Wilcox
<u>The Anatomy of Mathematics</u>, p. 77

* * *

Whether we regard mathematics from the utilitarian point of view, according to which the pupil is to gain facility in using a powerful tool, or from the purely logical aspect, according to which he is to gain the power of logical inference, it is clear that the chief end of mathematical study must be to make the pupil <u>think</u>. If mathematical teaching fails to do this, it fails altogether.

--John Wesley Young
<u>Lectures on Fundamental Concepts of Algebra and Geometry</u> (1911) p. 4

Calling All Cartographers

After one day spent on Glox, with a group of 13 year olds, an exercise on maps was developed. Sketches similar to the ones shown here were placed on the board. The students were asked: "Could this be a complete map of the region on Glox about which the spaceman sent messages?" This could be used as homework or adapted for a written exercise in class. Adaptations may need to be made for older children.

*

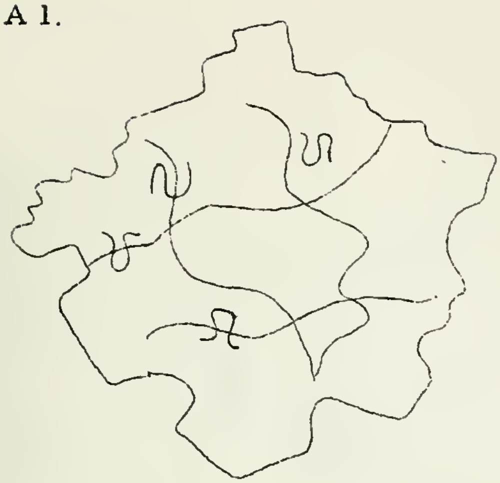
Long before the first space ship reached Glox, the dark side of the moon had been explored. Evidence had been found that intelligent beings had established a base there. Among the debris, papers had been discovered. Copies of these papers are shown on the next two pages. These pages have been labeled for easier identification. The drawing labeled A6 received a lot of attention. Everyone was certain that these were maps. But none of the others were recognizable.

When the messages started coming from the spaceman on Glox, the Commander set everyone the job of studying these maps to see if any of them were maps of the regions where the spaceman had landed on Glox. The first message made them discard A1 as a possible map. Why? Decide which of the others might be maps for that region of Glox. Justify your decisions.

Key

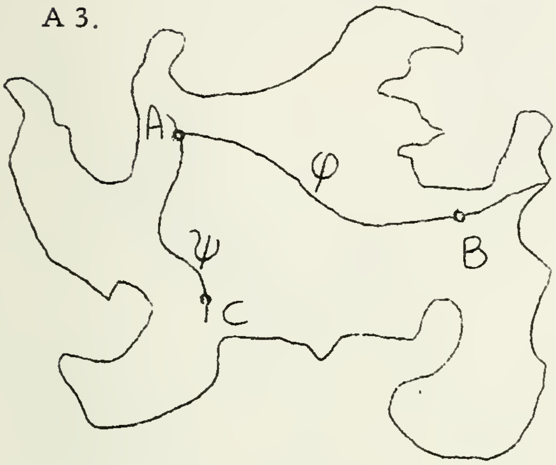
Map	Answer	Justification	
A1	No	Message 1 or 3	
A2	No	Message 3	
A3	No	Message 5	
A4	No	Message 2	
A5	No	Message 4	
A6	-	--------	
B1	No	Message 5	Two highways between C and D
B2	Yes		
B3	No	Message 3	
B4	No	Message 3	
B5	Yes		
B6	Yes		

A 1.



A 2.

A 3.



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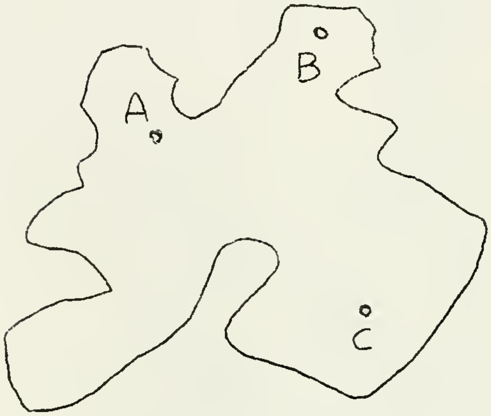

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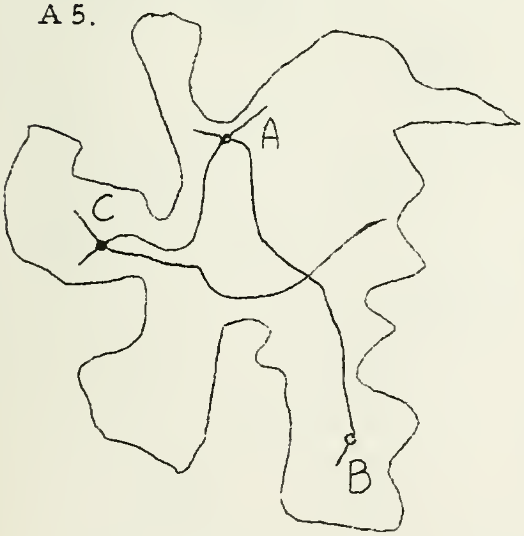
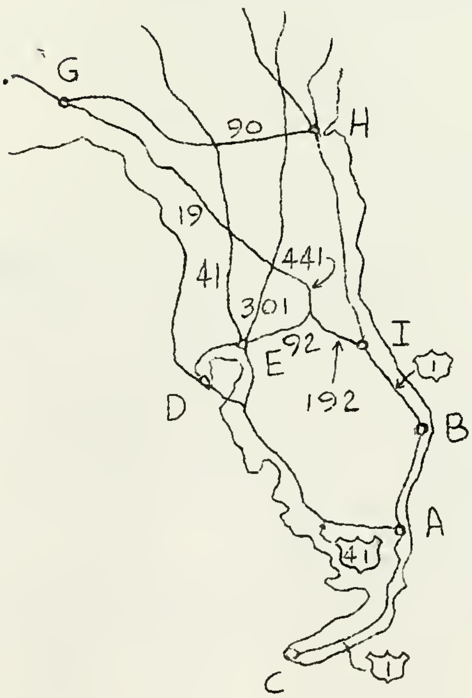
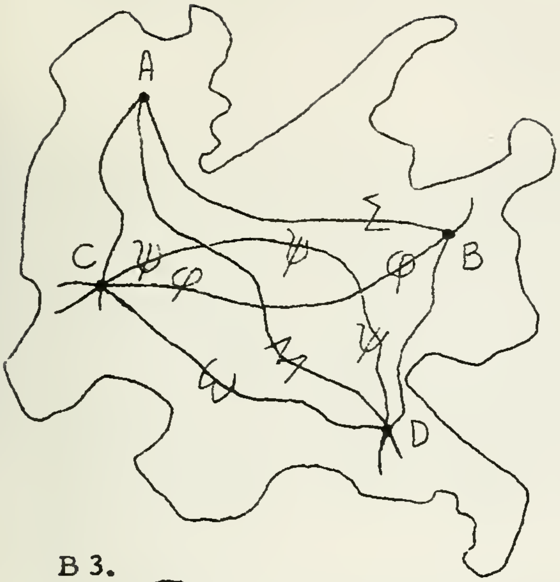


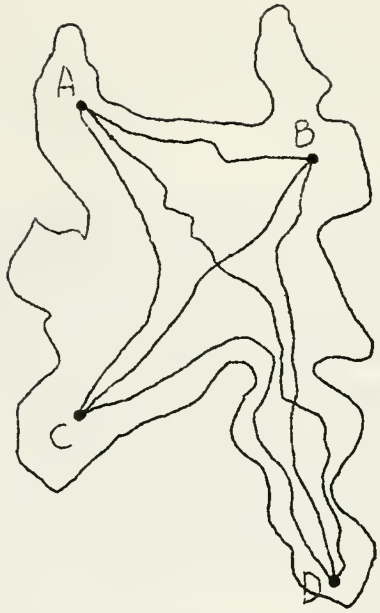
A 6.



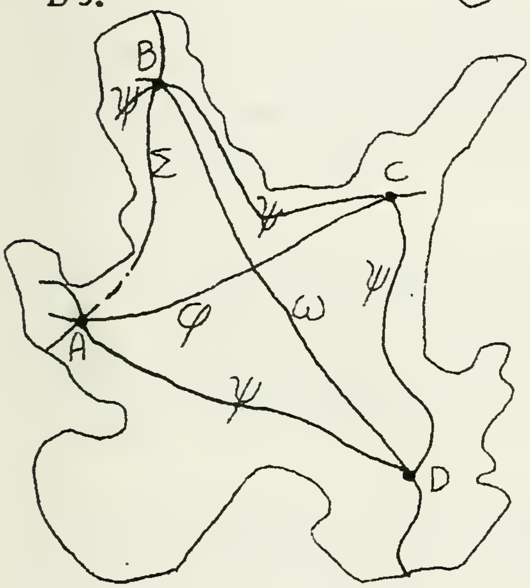
B 1.



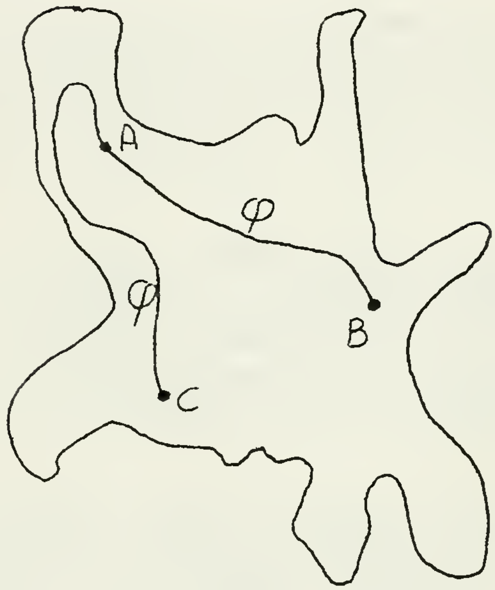
B 2.



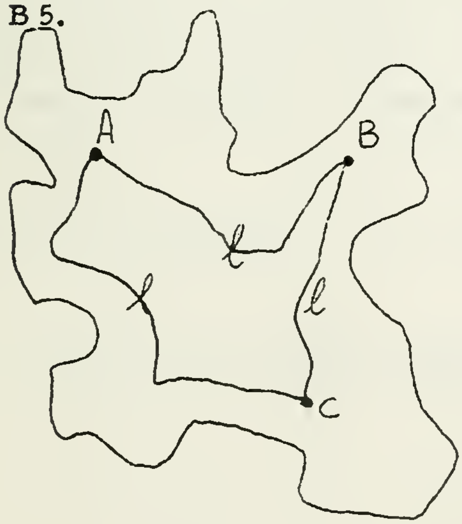
B 3.



B 4.



B 5.



B 6.

A Report on the Use of UICSM First Course Materials in Grades 7 and 8

(Courtesy of Miss McCoy)

In 1956-57, a class at the University [of Illinois] High School which was composed of superior 7th grade students was taught Units 1-3 of the UICSM First Course. The success of this experiment made it seem feasible to try these units with selected classes of 8th grade students in two of the Pilot schools. So, in 1957-58, two classes of 8th grade students at the Hough Street School [Barrington, Illinois] and one 8th grade class at the Beaver Country Day School [Chestnut Hill, Massachusetts] studied UICSM Units 1-3 instead of the usual eighth grade arithmetic course. The teachers, students, and principals of these schools were not only pleased, but enthusiastic, about the results of this experimentation; as a result, these two schools have continued, in succeeding years, to use Units 1-3 for their superior 8th grade students.

In 1958-59, 15 other schools began using UICSM First Course materials at the 8th grade level [in the Thomas Williams Junior High, Cheltenham, Pa. district, 1 class of 7th grade students was started in First Course]. In these schools, together with Hough Street, Beaver Country Day, and University of Illinois High, there was a total of 27 classes studying Units 1-3 at the 8th [or 7th] grade level.

In 1959-60, there were 27 of our cooperating schools using Units 1-3 for 8th [or 7th] grade students, with a total of 45 classes. The present school year, 1960-61, shows a big increase in the number of our cooperating schools which are offering the UICSM material for 8th grade students; there are now 51 such schools, and in these there are 95 classes. These 95 8th grade classes comprise approximately 31% of the First Course classes now being taught in our Cooperating Schools.

It should be pointed out that teachers using Units 1-3 at the 7th or 8th grade level are advised to supplement the text, when it seems necessary, with work on per cents, and on intuitive geometry. --The Project staff advises schools to make careful selection of students for 7th or 8th grade classes in this accelerated program; we recommend that students chosen have IQ scores about 120 or above, and an arithmetic achievement score of 8.5 or above [the latter is essential, since many of the discovery exercises in Units 1-3 presuppose an ability to compute].

Students who study Units 1-3 in the 8th grade usually study Units 4 and 5 in their second year of mathematics; they are then ready for Unit 6-Geometry-at the 10th grade.

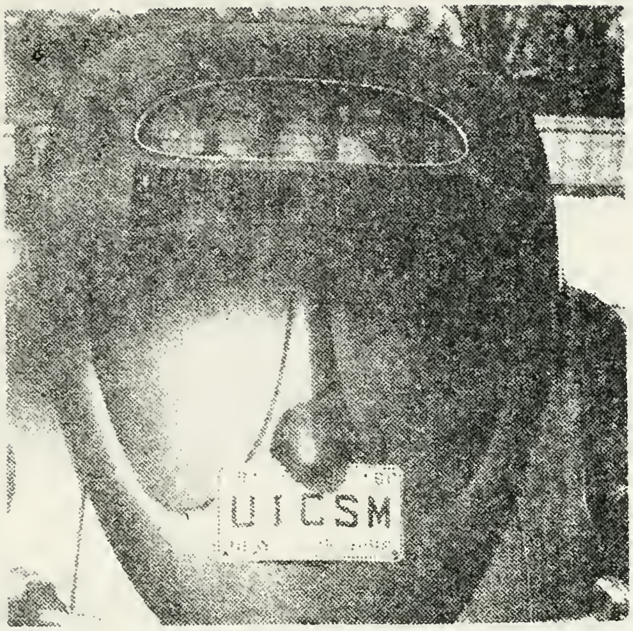
A comparison of results on our Unit tests indicates that the 8th grade students made scores as high, or higher, than did the students in 9th grade classes. A summary of test results is given on the next page.

Unit 1	1958-59		1959-60	
	7th-8th	9th	7th-8th	9th
*No. of students	560	2221	721	2581
Highest score	24	24	24	24
Third quartile	21	20	20	20
Median	18	18	17	17
First quartile	15	14	14	13
Lowest score	3	1	3	0
Mean	17.14	16.84	16.84	16.10
Standard deviation	4.50	4.56	4.34	4.90

Unit 2	1958-59		1959-60	
	7th-8th	9th	7th-8th	9th
*No. of students	539	2081	1069	2719
Highest score	24	24	24	24
Third quartile	19	18	18	18
Median	17	15	16	15
First quartile	14	12	12	11
Lowest score	4	2	4	2
Mean	16.24	14.97	15.1	14.4
Standard deviation	4.30	4.57	4.4	4.5

Unit 3	1958-59		1959-60	
	7th-8th	9th	7th-8th	9th
*No. of students	271	1888	609	2571
Highest score	24	24	24	24
Third quartile	18	17	18	17
Median	14	14	15	14
First quartile	12	11	12	11
Lowest score	4	2	3	1
Mean	14.37	14.00	14.91	13.99
Standard deviation	4.33	4.29	4.38	4.56

*The number of students whose scores have been tabulated shows variation from unit to unit due to the fact that our office had not received test results from all teachers [or even from the same number of teachers] by the time it seemed advisable to tabulate results.



Neither Numbers nor Numerals ?

Mr. Nathaniel Merrill, of the Newton South High School, Newton Center, Mass., sent in this photograph of a Volkswagen bearing 'UICSM' registration tags. We hope that there is some other automobile in New Hampshire inscribed 'SMSG', and that it is not a Cadillac.

A UICSM cooperating school, Pueblo High School in Tucson, Arizona, is one of three schools identified by journalist Martin Mayer as doing a surprisingly good job in spite of some very unfavorable socio-economic circumstances. Mr. Mayer's article, 'The Good Slum Schools', is drawn from his recent book, The Schools, and appeared in the April, 1961, Harper's Magazine.

Mr. Christ Kristo, of the Owatonna (Minnesota) High School, spoke before about 40 Sisters from nine elementary schools there recently in connection with UICSM.

Sister Mary of the Angels, St. Rosalia High School, Pittsburgh, spoke to the Diocesan Board of Supervisors May 5 on the need for curriculum change in the Junior and senior high schools.

Mrs. Mary Huzzard, Cheltenham Senior High School, Wyncote, Pa., went to Easton, Pa., on April 11 to talk to their mathematics teachers about the UICSM materials and Cheltenham's experience with them. She mentions having been questioned on the extent of her students' knowledge of analytic geometry by a college professor who was present, and that he seemed not to believe her students capable of what she reported.

Miss Olive V. Hicks, Desert Sun School, Idyllwild, California, was a member of the Reaction Panel at a University of Redlands conference on "Current Efforts to Improve Instruction in Mathematics". She took part in the Secondary Section meeting, devoted to "Mathematics Content Proposed for the Secondary School".

Mr. Eugene Epperson, Talawanda High School, Oxford, Ohio, reports a successful open house for parents at which he showed slides and talked about the purpose, history, and nature of the UICSM program. He also spoke at Miami University (Ohio) on April 27, to students and teachers in the area who had been invited to the campus for the day.

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UICSM Staff notes . . . Mr. Beberman took part in a mathematics curriculum conference at Boston University on April 26, the annual panel luncheon meeting of the Association of Teachers of Mathematics of New York City on April 29, and a secondary school mathematics workshop in Philadelphia on May 6; he also visited UICSM schools in New York State, New Jersey, Massachusetts, and Illinois early in May. ... Miss McCoy visited UICSM schools in Indiana, Arkansas, Missouri, Ohio, Pennsylvania, and West Virginia during April and May. ... Mr. O. Robert Brown spoke at the Sangamon County Teacher's Institute in Springfield, Illinois, on March 24.

* * *

Everything that I say really amounts to this, that one can know a proof thoroughly and follow it step by step, and yet at the same time not <u>understand</u> what it was that was proved.

And this in turn is connected with the fact that one can form a mathematical proposition in a grammatically correct way without understanding its meaning.

Now when does one understand it?--I believe: when one can apply it.

--Ludwig Wittgenstein
<u>Remarks on the Foundations of Mathematics</u> IV-25

UICSM Newsletter

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UNIVERSITY OF ILLINOIS COMMITTEE ON SCHOOL MATHEMATICS

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HELLO AGAIN

This newsletter is the sixth in a series of publications initiated by UICSM at the beginning of the preceding academic year. We shall continue to issue newsletters from time to time during 1961-62. Their primary purpose remains the same: to provide a supplement to and amplification of the Teacher's Commentaries for the various text units in the UICSM program. Other items of possible interest to teachers using UICSM materials will be included, such as test items and news of the professional activities of UICSM teachers and staff.

We assume that readers of this newsletter are somewhat familiar with our pedagogical methods and mathematical viewpoint as exemplified in the UICSM text materials. It is not possible to gain such familiarity by a casual reading of this newsletter. [For example, the third article in this issue contains some rather sophisticated language that is <u>not</u> to be found in the student texts. A reader not familiar with the student text might not realize this.]

Everyone who teaches or administers the UICSM program is again cordially requested to contribute material to our publication by submitting (to the Editor) anything he would like to share with others similarly occupied. We also want to hear about your professional activities in connection with or on behalf of UICSM, where you first encountered or have been studying the UICSM program, etc.

Please note that this newsletter is sent only to those whom we have reason to think are interested in the work of UICSM, whether because they are teaching from our materials or otherwise. If you wish to continue receiving it, please send to the project office your name, address, and a request that you be added to or retained on our mailing list for 1961-62. The form on the last page of this issue may be used for this purpose, as well as for sending us the names and addresses of others whom you feel should be receiving this publication. --Ed.

HOW TO TEACH UNITS 1 THROUGH 4
IN ONE YEAR

We are ready to admit that the complete four-year UICSM program will take more than four years to teach if all teaching is done in the recommended UICSM "style". If a school begins the UICSM program in grade 9 for unselected college preparatory students and wishes to take at least some of these students through the four-year program to the point where they will be able to pursue a college freshman course in analytic geometry and calculus, it will be necessary to de-emphasize certain topics and to speed up the presentation of some sections. Here are some suggestions for time allocations and teaching procedures for Units 1-4 that will enable a class to cover this material in one school year.

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Unit 1: The Arithmetic of the Real Numbers

The nature of this unit is such that no topic can safely be omitted. Each topic is important either for its own sake or because it provides important ground work for topics in later units.

The number-numeral problem should take one class lesson plus one assignment, the assignment being parts C, D, and E on pages 1-N and 1-O. Discussion of the Stan-Al correspondence should lead quickly to the point that in teaching somebody mathematics, you must acquaint him with mathematical entities such as numbers. In order to talk about these entities, you need names for them--for example, numerals. So, another part of teaching mathematics is acquainting learners with names. In order to talk about these names, you need names for them--for example, names of numerals. It is for this purpose--forming names of names--that single quotation marks are used.

Take at most 3 lessons for pages 1-1 through 1-15. You should be finished with page 1-23 at the end of the fifth lesson.

One secret in making rapid progress is to devote a good chunk of the class hour to "supervised study". An incredible amount of class time is spent in compelling students to listen to explanations and to participate in redevelopments of topics required by only a few students.

One lesson plus one assignment will take care of pages 1-24 through 1-28. Let students read pages 1-29 through 1-32 at their seats, you summarize the main point for them, zip through parts A and B on page 1-32, have five minutes of fun in class with part C on page 1-33, present the problem of ambiguity as discussed in the lower third of page 1-34, and use supervised study for students to read pages 1-33 through 1-35, plus homework on page 1-36.

Another notorious time-consumer is the checking of homework in class by reading answers and discussing points of disagreement. Why not duplicate a list of answers and pass this out either when checking time comes or, perhaps, when the assignment is made? For example, you might let students get started on part A on page 1-36 and pass out the answer key after three minutes of work. Of course, you would probably omit the "answer" to exercise 34. (You might even prefer to give the answers for just the even-numbered exercises even though you assigned all of them.)

Pages 1-37 through 1-42 are worth two lessons. And so are pages 1-42 through 1-49. One lesson can take care of 1-50 through 1-54 plus homework: part A on 1-55 and Supplementary Exercises, part G.

Pages 1-56 through 1-59 are critical. Spend two or three lessons on this with much class discussion.

Pages 1-60 through 1-66 can be handled in one supervised study lesson plus one homework assignment.

Pages 1-66 through 1-72 are critical. Spend two lessons on this.

Three lessons including a lot of supervised study will take care of pages 1-73 through 1-79.

Three more lessons on 1-80 through 1-85 with a lot of class discussion on 1-83 and 1-84.

One solid supervised study lesson plus homework will take care of 1-86 through 1-92. Another day will bring you through the middle of page 1-95.

Take another day for mixed drill work on operations.

Three lessons are necessary for the important work on pages 1-95 through 1-102.

Two more lessons will do it for 1-103 through 1-110.

The above schedule calls for some 29 lessons. If you throw in time for tests and review work (but not too many assemblies), you can complete Unit 1 in 7 weeks.

*

Unit 2: Generalizations and Algebraic Manipulation

As noted in the introduction to the commentary for Unit 2, the purpose of this unit is twofold: proving theorems, and applying these theorems to the manipulation of algebraic expressions. Just as it is unreasonable to expect complete mastery of manipulation in the space of one unit, so is it unreasonable to expect students to write the proofs of all the theorems in the unit. The important thing is that each student come to realize that any manipulation maneuver can be justified deductively--that it is a consequence of the basic principles of the arithmetic of the real numbers.

Pages 2-A through 2-I: one lesson plus one homework assignment, with 10 minutes of a second lesson on pages 2-G through 2-I.

Pages 2-1 through 2-7: one supervised study lesson plus one assignment. A bit of discussion on exercise 2, page 2-3; No. 6 on 2-4; No. 3 on 2-6.

Pages 2-8 through 2-13: two lessons plus two assignments.

Pages 2-14 through 2-23: three lessons plus three assignments. Read 2-14 through 2-16 in class. Let students read 2-17 and 2-18 by themselves. Supervise their work on page 2-19. At least half of the work on page 2-21 should be done under your supervision. This material is crucial.

Pages 2-23 through 2-27: one lesson plus one assignment. This is extremely important material, especially page 2-27.

Pages 2-28 and 2-29: optional.

Pages 2-30 through 2-38: three lessons plus homework. Critical material, calling for masterful teaching. This is the point at which students get a chance to see the purpose of proof and to use free variables in test-patterns.

Pages 2-39 through 2-44: one lesson plus homework.

Pages 2-45 through 2-59: four lessons plus homework, including supervised study.

Pages 2-60 and 2-61: one lesson plus homework (which should be carefully checked).

Pages 2-62 through 2-66: two lessons plus homework. Parts A, B, and C can be done orally in class. Much careful teaching must be done with pages 2-64 through 2-66, for it is here that students get their first experience in proving conditionals.

Pages 2-67 through 2-76: three lessons plus homework, including much supervised study over pages 2-71 through 2-75. Students should know what the theorems on pages 2-69 and 2-70 are about. Each student should be able to prove at least one theorem from part A and one theorem from part B.

Pages 2-77 through 2-80 (including the relevant Supplementary Exercises): two supervised study periods plus homework. Some of this could be blended with the work on pages 2-72 through 2-74.

Pages 2-81 through 2-84: one lesson plus homework from the Miscellaneous Exercises or Supplementary Exercises.

Pages 2-85 through 2-91: three lessons plus homework.

Pages 2-92 through 2-108: eight lessons plus homework, including much supervised study on drill lessons. Proofs of theorems need not be assigned as homework except as optional work, but some proofs should be presented at the blackboard.

Pages 2-109 through 2-111: one lesson plus homework.

Pages 2-112 through 2-158: six lessons. Some of this work should be distributed over the entire unit.

This schedule calls for some 40 lessons.

*

Unit 3: Equations and Inequations

There are blocks of material in this unit that should not be taught as units, but should have other materials interspersed within them. This is the case for the work on pages 3-40 through 3-50, 3-58 through 3-82, and 3-174 through 3-185. Such programing is not indicated in the schedule suggested below.

This unit offers many opportunities for supervised study, and many opportunities for time-wasting activities. Unless you consciously try to avoid it, you will find yourself trapped into forcing the entire class to listen to your explanation of an equation or a word problem which troubled only a few students.

Pages 3-A through 3-3: one lesson.

Page 3-4: optional.

Pages 3-5 through 3-10: two lessons.

Pages 3-11 through 3-18: three lessons with much supervised study.

Pages 3-19 through 3-21: one lesson.

Pages 3-22 through 3-25: one lesson.

Pages 3-26 through 3-32: three lessons with much class discussion.

Pages 3-32 through 3-50: five lessons with much supervised study.

Pages 3-51 through 3-57: three lessons including Supplementary Exercises on page 3-173.

Pages 3-58 through 3-82: eight lessons including Supplementary Exercises on pages 3-174 through 3-185.

Pages 3-83 through 3-95: four lessons.

Pages 3-96 through 3-99: two lessons.

Pages 3-100 through 3-107: four lessons.

Pages 3-108 through 3-111: two lessons.

Pages 3-112 through 3-120: two lessons, mostly supervised study.

Pages 3-121 through 3-123: one lesson plus homework, mostly supervised study.

Pages 3-127 through 3-130: one lesson.

Pages 3-131 through 3-133: one lesson.

Pages 3-134 through 3-136: two lessons.

Pages 3-137 through 3-197: eight lessons.

This schedule calls for some 54 lessons.

*

Unit 4: Ordered Pairs and Graphs

This unit has many potentialities for enrichment. It is particularly distressing to pare it down to a minimum course, but...

Pages 4-A through 4-H: three lessons, one of which should be spent on parts D and E.

Pages 4-1 through 4-4: one lesson.

Pages 4-4 through 4-10: two lessons.

Pages 4-11 through 4-19: three lessons.

Pages 4-20 through 4-25: one lesson [part B optional].

Pages 4-26 through 4-35: seven lessons.

Pages 4-36 through 4-41: one lesson.

Pages 4-42 through 4-47: two lessons.

Pages 4-49 through 4-51: one lesson.

Pages 4-52 through 4-55: one lesson.

Pages 4-56 through 4-58: two lessons.

Pages 4-59 through 4-64: three lessons, including Supplementary Exercises.

Pages 4-65 through 4-70: three lessons.

Pages 4-71 through 4-86: seven lessons.

Pages 4-87 through 4-90: one lesson [exercises 11 and 12 optional].

Pages 4-91 through 4-95: three lessons.

Pages 4-95 through 4-131: six lessons.

This schedule calls for some 47 lessons.

*

So, by following the schedule suggested above as closely as possible, you should be able to cover all four units in about 170 lessons. This does not include time for testing, assemblies, field trips, etc. Class time can be conserved by making some tests of the take-home variety. And, in order to gain more time, you might consider forming a faculty committee on decreasing the number of assemblies! -- M. B.

* * *

Four more Tests from the film study series are reprinted below in their original versions. Those designated A, B, C, and D were in Newsletter No. 1, E through H in No. 2, and I through L in No. 4. The page of the text to have been completed before the test is given is indicated in brackets. Hopefully these items will be of some interest and value to teachers when they begin teaching Unit 3. --R. S.

Test M [3-18]

Matching. <u>Remember</u>: an answer may be used once, or twice, or not at all.

Use these choices for questions 1 through 4:

(A) $\{x:\ xx \geq 1\}$ (B) $\{x:\ xx = x\}$ (C) $\{x:\ xx - 1 = 0\}$

(D) $\{x:\ |x| \leq 1\}$ (E) none of these

1. $\overline{-1,\ 1} =$ 2. $\overrightarrow{1,\ -1} =$ 3. $\{0,\ 1\} =$ 4. $\{-1,\ 1\} =$

Use these choices for questions 5 through 8:

(A) $\emptyset$ (B) $\{5\}$ (C) $\{5,\ -5\}$

(D) $\{x:\ x + 5 = x + 5\}$ (E) none of these

5. $\{x:\ |x| = 5\} =$ 6. $\{x:\ xx + 5 > 0\} =$

7. $\{t:\ t(-t) = -25\} =$ 8. $\{x:\ x + 5 = x\} =$

Use these choices for questions 9 through 12:

(A) $\overline{-1,\ 1}$ (B) $\overline{-1,\ 1}$ (C) $\overrightarrow{-1,\ 1}$

(D) $\overrightarrow{-1,\ 1}$ (E) none of these

9. $\{x:\ |x| < 1\} =$ 10. $\{x:\ -1 \leq x < 1\} =$

11. $\{x:\ x \geq 1 \text{ and } x \leq -1\} =$ 12. $\{x:\ xx < 1 \text{ or } x = -1\} =$

Choose the one correct answer.

13. The midpoint of a number line segment is 2 and one end point is 10. What is the other end point?

(A) -6 (B) 6 (C) 8 (D) 18 (E) none of these

14. The end points of one number line segment are −2 and 6. The end points of another are 4 and 16. The midpoint of the segment joining their midpoints is

(A) 4 (B) 5 (C) 6 (D) 7 (E) none of these

15. For each x, if the end points of a segment are $(4x + 7)$ and $(2x - 3)$, the midpoint is

(A) $2x + 10$ (B) $3x + 2$ (C) $6x + 4$ (D) $8x - 21$ (E) none of these

16. Which of these sets contains no elements?

(A) $\{a: -aa > 0\}$ (B) $\{\emptyset\}$ (C) $\{b: 7b + 5b = 11b\}$

(D) $\{0\}$ (E) none of these

17. Look at these names of number line loci:

$\{x: x \geq 1\}$ $\{y: |y| = 1\}$ $\{s: ss > 0\}$

$\{m: m < 0 \text{ or } m > 1\}$ $\overrightarrow{0, 1}$ $\overrightarrow{1, 0}$

There are, among the loci named, exactly

(A) no intervals and no segments

(B) no intervals and one segment

(C) one interval and no segments

(D) one interval and one segment

(E) none of these

18. Look at these names of number line loci:

$\overrightarrow{1, 0}$ $\{z: -1 \leq z < 0\}$ $\{q: |q| = 1\}$

$\{d: -d \leq 0\}$ $\overrightarrow{1, 1}$ $\{0\}$

There are, among the loci named, exactly

(A) no half-lines and no rays

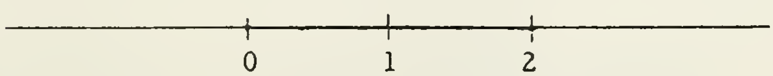
(B) no half-lines and one ray

(C) one half-line and no rays

(D) one half-line and one ray

(E) none of these

19. Which of the following is a name for this number line graph?



(A) $\overline{0, 2}$ (B) $\overrightarrow{0, 2}$ (C) $\{x: |x| \leq 2\}$

(D) $\{x: x \geq 0 \text{ and } x \leq 2\}$ (E) none of these

20. Which of the following is a name for this number line graph?

(A) $\{w: ww \leq 1\}$ (B) $\{x: |x| < 1\}$ (C) $\{f: -1 < f \leq 1\}$

(D) $\overline{-1, 1}$ (E) none of these

Key for Test M [3-18]:

1. D	2. E	3. B	4. C	5. C	6. D	7. C
8. A	9. E	10. B	11. E	12. B	13. A	14. C
15. B	16. A	17. A	18. E	19. D	20. C	

Test N [3-31]

In each of problems 1 through 10, two sets P and Q are described.
Mark A if $P \subseteq Q$ and $Q \subseteq P$,
B if $P \subseteq Q$ and $Q \not\subseteq P$,
C if $Q \subseteq P$ and $P \not\subseteq Q$,
D if $P \not\subseteq Q$ and $Q \not\subseteq P$.

1. $P = \{x: |x| = 1\}$; $Q = \{x: xx = 1\}$

2. $P = \{x: |x - 4| = 1\}$; $Q = \{x: xx + 3 = 4x\}$

3. $P = \{x: xx = x\}$; $Q = \{s: 3s + 4s = 5s\}$

4. $P = \{y: 15y - 7 = 5 + 9y\}$; $Q = \{0, 2\}$

5. $P = \{f: 5f + 3 - f = 13f - 9f + 3\}$; $Q = \{g: 2g = 3(5g - 4g)\}$

6. $P = \{m: 7m - 3 = 89\}$; $Q = \{m: (7m - 3) + 3 = 92\}$

7. $P = \{n: 5(2n - 3) = (17 - n)5\}$; $Q = \{n: 2n - 3 = 17 - n\}$

8. $P = \{x: (x + 7)x = 5x\}$; $Q = \{y: y + 7 = 5\}$

9. $P = \{z: 81 - 3z = 78\}$; $Q = \{v: 81 = 78 + 3v\}$

10. $P = \{x: (x + 7)(x - 5) = (x + 7)(x + 2)\}$; $Q = \{x: 7(x - 5) = 7(x + 2)\}$

Which is the correct solution set, if listed, for these equations?

11. $x + (x + 2) + (x + 4) = 27$

(A) $\emptyset$ (B) $\{7\}$ (C) $\{21\}$ (D) none of these

12. $|3 - 2m| = 1$

(A) $\{1\}$ (B) $\{2\}$ (C) $\{1, 2\}$ (D) none of these

13. $xx + 1 = 0$

(A) $\emptyset$ (B) $\{1\}$ (C) $\{-1\}$ (D) none of these

14. $2 = 3 - 9yy$

(A) $\emptyset$ (B) $\{\frac{1}{3}\}$ (C) $\{\frac{1}{9}\}$ (D) none of these

15. $\frac{1}{5}(5z - 15) + \frac{1}{3}(9z + 12) = 25$

(A) $\{5\}$ (B) $\{6\}$ (C) $\{7\}$ (D) none of these

16. $6x - 5 + 3x(2 - x) - 4x(3 - x) = 116$

(A) $\{-11\}$ (B) $\{-11, 11\}$ (C) $\{121\}$ (D) none of these

III. Choose the one correct answer.

17. P and Q are sets. If P is a subset of Q and 3 is not a member of Q, what follows?

(A) $\{3\}$ is a subset of P

(B) there is a member of Q which is not in P

(C) 3 is not a member of P

(D) none of these

18. How many subsets of $\{1, 2\}$ are there?

(A) 4 (B) 3 (C) 2 (D) none of these

19. What is needed to show that $\{x: 3x + 5 = 17\} \subseteq \{x: 3x = 12\}$?

(A) $\forall_x \forall_y \forall_z$ if $x = y$ then $x + z = y + z$

(B) $\forall_x \forall_y \forall_z$ if $x + z = y + z$ then $x = y$

(C) both (A) and (B)

(D) none of these

20. Why is $\{y \neq 0: \frac{10}{y} + 3 = 8\} = \{y \neq 0: (\frac{10}{y} + 3)y = 8y\}$?

(A) $\forall_x \forall_y \forall_{z \neq 0}$ if $x = y$ then $xz = yz$

(B) $\forall_x \forall_y \forall_{z \neq 0}$ if $xz = yz$ then $x = y$

(C) both (A) and (B)

(D) none of these

Key for Test N [3-31]:

1. A	2. D	3. C	4. B	5. C	6. A	7. A
8. C	9. A	10. C	11. B	12. C	13. A	14. D
15. B	16. B	17. C	18. A	19. B	20. C	

Test O [3-57]

I. For each pair of equations below, choose the most immediate justification for the step from the first equation to the second. Mark

A if the justification is 'equivalent expressions',

B if the justification is 'the addition transformation principle',

C if the justification is 'the multiplication transformation principle', or

D if none of these justifications apply.

1. $$\left.\begin{aligned} 5x - 13 &= 8 - 7x \\ 5x - 7x - 13 &= 8 \end{aligned}\right\}$$

2. $$\left.\begin{aligned} \frac{a}{3} - 7 &= 2 + \frac{a}{4} \\ \left(\frac{a}{3} - 7\right)12 &= \left(2 + \frac{a}{4}\right)12 \end{aligned}\right\}$$

3. $$\left.\begin{aligned} 5y - 3 &= 9 + 2y \\ 5y - 3 + 3 &= 9 + 2y + 3 \end{aligned}\right\}$$

4. $$\left.\begin{aligned} \frac{5}{x} &= 4 - \frac{3x - 5}{x} \\ \left(\frac{5}{x}\right)x &= \left(4 - \frac{3x - 5}{x}\right)x \end{aligned}\right\} [x \neq 0]$$

5. $$\left.\begin{aligned} \frac{y - 7}{y + 3} + \frac{2}{y + 3} &= 8 \\ \left(\frac{y - 7}{y + 3} + \frac{2}{y + 3}\right)(y + 3) &= 8(y + 3) \end{aligned}\right\} [y \neq -3]$$

II. Choose the correct solution set for each of the following, or mark (D) if none are correct.

6. $xx = 3x$

(A) $\emptyset$ (B) $\{3\}$ (C) $\{0, 3\}$ (D) none of these

7. $\frac{x}{3} + 5 = x + \frac{19}{3}$

(A) $\{1\}$ (B) $\{\frac{1}{2}\}$ (C) $\{-2\}$ (D) none of these

8. $\frac{12}{3x - 7} = \frac{3}{7 - x}$

(A) $\{5\}$ (B) $\{7\}$ (C) $\left\{\frac{85}{21}\right\}$ (D) none of these

9. $\frac{4}{x + 6} = \frac{8}{x - 6}$

(A) $\{-3\}$ (B) $\{6\}$ (C) $\{18\}$ (D) none of these

10. $5(x - 7) = (4 - 3x)(7 - x)$

(A) $\emptyset$ (B) $\{3\}$ (C) $\{7\}$ (D) none of these

II. Choose the correct result when each of the following equations is solved for 'x'.

11. $ax + by + c = 0$

(A) $x = \frac{c - by}{a}$, $[a \neq 0]$ (B) $x = \frac{by - c}{a}$, $[a \neq 0]$

(C) $x = -\frac{by + c}{a}$, $[a \neq 0]$ (D) none of these

12. $a + bx = x + c$

(A) $x = \frac{c - a}{1 - b}$, $[b \neq 1]$ (B) $x = \frac{c - a}{b - 1}$, $[b \neq 1]$

(C) $x = \frac{a + bx}{c}$, $[c \neq 0]$ (D) none of these

13. $\frac{1}{x} + \frac{1}{y} = 5$

(A) $x = \frac{1 - 5y}{5}$ (B) $x = y(5x - 1)$

(C) $x = \frac{y}{1 - 5y}$, $[y \neq \frac{1}{5}]$ (D) none of these

14. $y = \frac{1}{x} - n$

(A) $x = \frac{n + y}{ny}$, $[ny \neq 0]$ (B) $x = \frac{1 - nx}{y}$, $[y \neq 0]$

(C) $x = \frac{1}{n + y}$, $[y \neq -n]$ (D) none of these

15. $A = \frac{Bx}{C + Ex}$

(A) $x = \frac{AC}{AE - B}$, $[AE \neq B]$ (B) $x = \frac{Bx - AC}{AE}$, $[AE \neq 0]$

(C) $x = \frac{AC}{AC - B}$, $[AC \neq B]$ (D) none of these

IV. Choose the correct solution set for each equation from among these:

(A) $\emptyset$ (B) $\{0\}$ (C) the set of all real numbers

Mark (D) if none of these is correct.

16. $x + 3 = x - 5$

17. $3x = 5x$

18. $3 + 4 = 5$

19. $(x + 3)(x - 3) = xx - 9$

20. $2 + 2 = 4$

Key for Test O [3-57]:

1. D	2. C	3. B	4. C	5. C	6. C	7. C
8. A	9. D	10. D	11. C	12. B	13. D	14. C
15. D	16. A	17. B	18. A	19. C	20. C	

Test P [3-82]

Choose the correct answer to each of the following problems. If the correct answer is not given, mark (D).

1. For each number x of arithmetic, x gallons of a 25% alcohol solution contain __?__ gallons of alcohol.

   (A) $\frac{x}{4}$ (B) 25x (C) $\frac{x}{.25}$ (D) none of these

2. A dog chasing a rabbit, which has a start of 150 feet, jumps 9 feet every time the rabbit jumps 7. The dog will overtake the rabbit in __?__ leaps.

   (A) 300 (B) 150 (C) 75 (D) none of these

3. A student has quiz scores of 100 and 84. What score must he achieve on a third quiz in order to have an average score of 90?

   (A) 86 (B) 94 (C) 96 (D) none of these

4. For each number J of arithmetic, for each nonzero number K of arithmetic, J is __?__ times as large as K.

   (A) JK (B) $\frac{K}{J}$ (C) $\frac{J}{K}$ (D) none of these

5. For each u, for each v, what must be subtracted from u to yield a difference of v?

   (A) u - v (B) v + u (C) v - u (D) none of these

6. For each number E of arithmetic, for each nonzero number F of arithmetic, E is __?__ % of F.

   (A) $\frac{100E}{F}$ (B) $\frac{E}{F}$ (C) $\frac{E}{100F}$ (D) none of these

7. For each number M of arithmetic, for each number K of arithmetic, if a train runs M miles in 5 hours, how many miles will it run in K hours at the same rate of speed?

   (A) $\frac{5M}{K}$ (B) $\frac{MK}{5}$ (C) 5MK (D) none of these

8. Charlie Brown is thinking of a number. If he multiplies it by 4 and then adds 5 to the product, he gets a sum of 12. What number is he thinking of?

   (A) $\frac{5}{3}$ (B) $\frac{3}{4}$ (C) 2 (D) none of these

9. Delores Haze is thinking of a number. If she subtracts 251 from it and then divides the difference by 7, she gets a quotient of 13. What number is she thinking of?

   (A) 342 (B) $252\frac{5}{7}$ (C) 160 (D) none of these

10. If you increase a certain number by 17, you get the same result as if you had subtracted half the number from 5. What is this number?

    (A) −6 (B) $-\frac{7}{3}$ (C) 8 (D) none of these

11. If a house and a lot together cost $22, 000 and the house costs $20, 000 more than the lot, the lot alone costs $__?__.

    (A) 2, 000 (B) 1, 500 (C) 1, 000 (D) none of these

12. In a certain group of 600 people, three-fifths of them are men. Four-fifths of the people in the group are married. There are __?__ married men in the group.

    (A) 360 (B) 240 (C) 288 (D) none of these

13. A rectangular flower garden is 20 feet longer than it is wide. Its perimeter is 360 feet. The length of the flower garden must then be __?__ feet.

(A) 160 (B) 80 (C) 100 (D) none of these

14. The quarterback of the Zabranchburg High football team has completed 7 out of 14 passes in a practice session. The coach says the squad will keep practicing until 65% of the passes are complete. The quarterback must throw at least __?__ more passes before practice is over.

(A) 8 (B) 6 (C) 14 (D) none of these

15. A hiker can average 2 miles per hour going uphill and 6 miles per hour going downhill. His average speed for a trip to the top of the hill and back, if he spends no time at the top, is __?__ miles per hour.

(A) 3 (B) 4 (C) 8 (D) none of these

16. For each x, if x is a number of arithmetic between 0 and 50, a man traveling 100 miles at x miles per hour arrives at his destination 2 hours late. How many miles per hour should he have traveled in order to arrive on time?

(A) $\frac{50 - x}{50x}$ (B) $\frac{50x}{50 - x}$ (C) $\frac{100 - 2x}{x}$ (D) none of these

17. A small pump can fill a pool in 6 hours. A larger pump can do it in 2 hours. Two small pumps and one larger one, working together, can fill the pool in __?__ hour(s).

(A) 4 (B) 1 (C) 1.2 (D) none of these

18. If the length of a rectangle is increased by 20% and its width increased by 10%, the area of the rectangle will be increased by __?__%.

(A) 15 (B) 20 (C) 30 (D) none of these

Key for Test P [3-82]:

1. A	2. C	3. A	4. C	5. A	6. A	7. B
8. D	9. A	10. D	11. C	12. C	13. C	14. B
15. A	16. B	17. C	18. D.			

A MATHEMATICAL DESCRIPTION OF UNITS 1 AND 2

As a starting point, we assume that the students are somewhat familiar with what we call the <u>system of numbers of arithmetic</u>. This system can be back-handedly described as the system of "unsigned" real numbers. It is isomorphic--with respect to ordering and the fundamental operations--to the system of nonnegative real numbers. The real numbers can be defined to be equivalence classes of ordered pairs of numbers of arithmetic. This is <u>not</u> done in the text; the possibility is mentioned in the COMMENTARY for page 1-1 of Unit 1, TC[1-1]a and b. Thus, in terms of concepts with which students become acquainted in Unit 5 (Introduction and 5.01), each real number is a relation among the numbers of arithmetic; for example, $^+2$ is the relation of being 2 greater than, and $^-\sqrt{3}$ is the relation of being $\sqrt{3}$ less than.

Although the authors may think of a number of arithmetic as a set-theoretic entity of a certain kind--a set of sets of ordered pairs of finite cardinal numbers--and of a real number as a set of ordered pairs of numbers of arithmetic, the familiarity with the numbers of arithmetic which may be expected of students has, of course, a different basis. A student's feeling of being acquainted with the numbers of arithmetic probably has its origin in his experiences of using numerals for these numbers to formulate accepted answers to questions concerning measures of magnitudes such as lengths, areas, weights, etc., and of manipulating such sumbols to obtain accepted answers to further questions of this kind. That these numerals, since they function as nouns, refer to entities of some kind, is an inference which, justifiably or not, he makes without much conscious thought. For him, numbers of arithmetic are things whose names occur in a characteristic way in sentences about measures of magnitudes.

This attitude toward the numbers of arithmetic makes a good starting point from which to develop a similar feeling for the real numbers. [See pages 1-1 through 1-4 and TC[1-2, 3].] For example, this morning the dollar-measure of the money in my pocket was 3.59, and it is now 3.22. By a physical process analogous to subtraction the magnitude of my cash-on-hand has decreased by a magnitude whose dollar-measure is 0.37. Speaking somewhat loosely, the magnitude of my solvency has undergone a <u>change</u> which involves both a magnitude and a direction. This change can be <u>described</u>, as above, by giving its direction and a measure of its magnitude. Can it, itself, be <u>measured</u>? It seems reasonable that changes in magnitude should be measurable, but that their measures will be numbers of some kind other than the numbers of arithmetic, which are measures of magnitudes. That there are such other numbers--the real numbers--becomes an article of faith, on the same level as the student's belief in the existence of the numbers of arithmetic. In particular, the dollar-measure of the change in the magnitude of my cash-on-hand is the real number $^-0.32$.

It is to be noted that, on its own level, the above method of calling attention to the real numbers parallels the author's underlying conception of the real numbers as relations among the numbers of arithmetic. Magnitudes are measured by numbers of arithmetic, and changes in magnitude--which can be identified with relations among magnitudes (for example, the relation of being 0.37 dollars less than)--are measured by relations among the numbers of arithmetic.

Throughout the course students are led to build new mathematical concepts from earlier ones by ways which, like the foregoing, parallel the relationships which the former have to the latter in the underlying mathematical philosophy adopted by UICSM. As a result of such an approach, students see the parts of mathematics they study as parts of a single subject rather than as a collection of somewhat disparate topics. Consequently they have a better understanding of what mathematics is, and develop more power and freedom to make use of what they learn.

Returning to the students' observations of real numbers, it is now easy to discover how to compute the measure of the change in a magnitude which is the result of two successive changes whose measures are known. [See pages 1-5 through 1-8 and TC[1-8]a, b.] This discovery focuses attention on a certain operation on real numbers and motivates practice in finding the result of applying this operation to given real numbers. Thus students first discover the operation of addition of real numbers and become convinced of its utility before deciding that it is reasonable to call this operation 'addition'. Among the advantages of this approach is the by-passing of certain difficulties which may arise when addition of real numbers is introduced by a definition which, in part, tells the student that in order to find the <u>sum</u> of a positive and a negative number he should begin by finding the <u>difference</u> of the two corresponding numbers of arithmetic.

A similar device, dealing with rates of change, draws the students' attention to another operation--multiplication of real numbers--and stimulates him to practice it. [See pages 1-17 through 1-22.] The procedures which students develop for computing sums and products of real numbers lead easily to an awareness of the isomorphism--as far as concerns addition and multiplication--between the system of the numbers of arithmetic and that of the nonnegative real numbers. Quoting the text [page 1-30]:

> the nonnegative real numbers <u>act like</u> the numbers of arithmetic with respect to <u>both</u> addition and multiplication.

The isomorphism just referred to suggests that it will often not be confusing to use the same symbol (for example, '3') either as a name for a number of arithmetic or as a name for the corresponding nonnegative real number (i.e., either as a name for the number 3 of arithmetic or as a name for the real number $^{+}3$).

Students' previous experience with the operations of addition and multiplication of numbers of arithmetic makes it easy for them to become aware of the commutative, associative, and distributive properties of these operations, and of the neutral character of the numbers 0 and 1. [See pages 1-44 through 1-48.] They see the value of being aware of these properties by discovering that they can be used to predict the successful outcome of a shortcut. [Page 1-56] This is also preparation for discovering proofs. The isomorphism previously noted suggests that addition and multiplication of real numbers may have similar commutative, associative, and distributive properties, a suggestion which is strengthened by carrying out some computations. [Pages 1-60, 61 and TC[1-60, 61]a, b] Such testing by computation prepares students to recognize, accept, and use instances of, for

example, the associative principle for multiplication, such as:

$$^{+}385.7 \cdot (^{-}56.2 \cdot \frac{^{+}5}{2}) = (^{+}385.7 \cdot {^{-}56.2}) \cdot \frac{^{+}5}{2}$$

without testing them.

At this point (in Unit 1) students are still working with only the arithmetic of the real numbers. They have not yet been introduced to variables and so have no efficient way of stating the principles, for example:

$$\forall_x \forall_y \forall_z \ x(yz) = (xy)z,$$

whose instances they now recognize and accept. This introduction comes early in their study of Unit 2, during the second quarter of their first UICSM year. By this time they have gained sufficient familiarity with instances of the principles to have little trouble in formulating the principles for themselves once the appropriate linguistic devices of variables and quantifiers have been introduced.

In the remainder of Unit 1 students use instances of these "basic principles for real numbers" at appropriate points to justify computational short cuts. This is valuable preparation for learning, as they do in Unit 2, to derive theorems from the basic principles. The interplay between the process of discovering real number properties through computation, and of formulating such discoveries and deriving the formulations from the basic principles, has much to do with the growth of the students' understanding of mathematics and of what mathematics is.

UICSM students are unlikely (to say the least) to adopt the usual lay viewpoint that the world's greatest mathematician is the <u>idiot-savant</u> who can perform the most astonishing feat of mental computation. The value of the interplay between discovery and proof is underlined by the first part of Hadamard's statement: "The object of mathematical rigor is to sanction and legitimatize the conquests of intuition, and there never was any other object for it."

Students are next led to conceive of a singulary operation, such as that of adding $^{-}5$ or of multiplying by $^{+}2$, as a set of ordered pairs. [Pages 1-67, 68, 75, 76] For example, one member of adding $^{-}5$ is the pair $(^{+}7, {^{+}2})$. Defining subtraction as the inverse of addition--in particular, subtracting $^{-}5$ is the operation that undoes what adding $^{-}5$ does--leads rapidly to recognition of the fact that the pairs belonging to a subtracting operation are the converses of those which belong to the corresponding adding operation. Example: because $(^{+}7, {^{+}2})$ belongs to adding $^{-}5$, $(^{+}2, {^{+}7})$ belongs to subtracting $^{-}5$. Moreover, it turns out, on comparison of their members, that the operation of subtracting $^{-}5$ is the same as the operation of adding $^{+}5$. In general, subtracting a real number is the same as adding its <u>opposite</u>. (In Unit 2, this is formulated as the <u>principle for subtraction</u>:

$$\forall_x \forall_y \ x - y = x + -y.$$

Together with the <u>principle of opposites</u>:

$$\forall_x \ x + -x = 0$$

and the previously mentioned principles concerning addition and multiplication it furnishes a basis from which to derive theorems about subtraction.)

Dividing by a nonzero real number is introduced as the inverse of multiplying by that number, and an exercise suggests that multiplying by 0 has no inverse. The further development of this insight is reserved for Unit 2.

The introduction of the basic concepts involved in the structure of the real number system as an ordered field is completed by defining the relation less-than. After doing this there is some practice with the notations '<', '>', '$\nless$', '$\geq$', etc. Solution of inequations is taken up in Unit 3 and the deductive organization of the theory of order is undertaken in Unit 7. However, in Unit 2, the use of '<', etc., significantly increases the variety of exercises which can be formulated. It also gives students an opportunity to become accustomed to considering inequations to be as "natural" as equations are.

The final topic in Unit 1 is the absolute value operation. For pedagogical reasons this operation is defined, here, as an operation from the real numbers to the numbers of arithmetic. Its converse is the union of the two operations "positiving" and "negativing", which map the numbers of arithmetic on the "corresponding" nonnegative and nonpositive numbers, respectively. The three operations--absolute valuing, positiving, and negativing--make it possible, in Unit 2, to formulate definitions of addition and multiplication of real numbers in terms of the corresponding operations on numbers of arithmetic and, in general, to pass readily back and forth from one system to the other. In later units we introduce the more usual use of 'absolute value' to denote a mapping of the real numbers on the nonnegative real numbers. This second mapping is, of course, the composition of the positiving operation with the original absolute value operation.

*

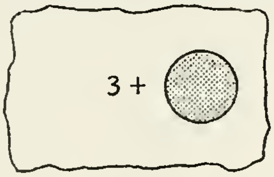
The purpose of Unit 2 is twofold: to help students become proficient in the elementary techniques of symbol-pushing used in simplifying algebraic expressions, and to lead them to discover, prove, and use the theorems about numbers which justify these techniques. The attainment of the first of these goals--mastery of the skills whose practice makes up the bulk of a traditional secondary school mathematics program--is, of course, a sine qua non for further progress in mathematics. Efforts expended toward the second goal have turned out, perhaps rather unexpectedly, to provide strong motivation for attaining the first. UICSM students of all degrees of mathematical aptitude have found the discovery and proof of theorems to be an exciting and rewarding experience. But, they also find that errors in manipulation place barriers in the way of discovering provable theorems!

Work toward either of these goals thus helps in attaining the other, and the attainment of either is easier for one who has a clear understanding of some of the purposes for which letters are used in mathematical language. Two of these are particularly relevant to the work of Unit 2: the use of letters as real (or: free) variables, and their use as apparent (or: bound) variables. For example, in the expression:

$$\int_1^x (t + 2y)^2 dt$$

the 'x' and the 'y' are free ("*real*ly variables") and the 't's are bound ("only *apparent*ly variables"). The 'x' marks a position that is open to substitution; the 't's do not, but can, without changing the meaning of the expression, be replaced by occurrences of any other "dummy" symbol. (Since the above use of 'real variable' conflicts with its conventional meaning in, say, 'theory of functions of a real variable', it is not introduced into the text. *However, it will be convenient to continue this usage in what follows.*)

Since a real variable merely marks a place where substitutions "can" be made (i.e., an argument-place) and, once its domain is specified, delimits the class of expressions that can be substituted for it, it turns out to be enlightening to simulate the notion of free variable by using actual holes in the paper. For example:



[See Unit 2, Introduction.] The notion of substitution is conveyed by writing a numeral on a second sheet of paper and placing this sheet under the first so that the numeral appears in the hole. The next step is to use frames ($\square$, $\triangle$, $\bigtriangledown$, $\boxed{}$, etc.) in which numerals and other appropriate expressions can be written. (Such expressions may themselves contain frames and so allow for further "substitutions".) Finally, letters are introduced as real variables. The first expressions using letters as variables are those on pages 2-19 through 2-23. Parts *D* and *E* help students to develop an understanding of algebraic "form".

An alternative concept often introduced in preference to the concept of a real variable is that of an "unknown". In line with this latter concept the 'x' in, say, '$x + 3 = 5$' is a numeral for a "definite, but unspecified" number. One advantage claimed for this concept is that it leads students to manipulate variables in the same way as they have learned to manipulate (other) numerals. This is, of course, a desirable end, and it should be obvious that the same result is attained when such occurrences of 'x' are construed as real variables. Clearly, a symbol for which numerals can be substituted fares, during manipulation, exactly as do numerals. While one is symbol-pushing he treats a real variable *as if* it were a numeral. The inadequacy of the concept of 'x' as an unknown is first apparent when one considers equations such as '$x + 1 = x$' and '$x^2 + x - 2 = 0$'. The "definite but unspecified" number which 'x' is supposed to represent fails, in the first case, to exist, and, in the second, to be unique. Students have some difficulty in extending their concept of number to include such queer entities. The concept of 'x' as an indeterminate is, of course, something entirely different. It seems unlikely that this concept, valuable as it is in modern algebra, would be helpful to students at this stage. A fourth concept, that of a random variable, i.e., a measurable function, is also beyond consideration at *this* level.

Since the word 'variable' has inappropriate connotations [see Note 1, below], it seems desirable to use at first a word more descriptive of the actual function of real variables. Now, it is a fact that real variables serve

one of the purposes for which pronouns are used: they hold places for nouns. [Note 2] Since the free variables used in Unit 2 have domains consisting of numbers, and since nouns for numbers are numerals, it is helpful, at this stage, to coin the word 'pronumeral' to denote numerical real variables. When, in later units, variables with non-numerical domains make their appearance, the concept of real variable is sufficiently well established that the word 'variable' can be introduced with no danger of misunderstanding.

The role that apparent variables play in mathematical language is also one played in other languages by pronouns--and, more frequently, by common nouns. (So, in Unit 2, apparent variables are also included among the pronumerals.) This role is that of linking operators with argument places. For example, the role of the 't's in:

$$\int_1^x (t + 2v)^2 dt$$

is to link the operator '$\int_1^x \quad d\ldots$' with the argument-place occupied by 'u' in the expression '$(u + 2v)^2$'. [Note 3] Natural languages are quite irregular in the operations they use and the ways in which these are linked with argument-places. However, the following somewhat archaic-sounding sentence illustrates how the operator (more specifically: the quantifier) 'each' is linked by means of a common noun and two pronouns to two argument-places:

Each man, as he is honorable, so shall he prosper.

In the same vein, the following sentence illustrates the linking of two quantifiers, each to its appropriate argument-places:

Each man, and each woman, as he cherishes her,
so shall she cleave to him.

In mathematical language constructions similar to those in the two examples above are used in the statements of two of the basic principles for real numbers:

For each x, $x + 0 = x$

and:

For each x, for each y, $xy = yx$.

About half-way through the text of Unit 2 (but usually earlier in the classroom) these principles are rewritten as:

$$\forall_x \; x + 0 = x \quad \text{and:} \quad \forall_x \forall_y \; xy = yx.$$

The discussion and exercises leading up to such use of apparent variables occur on pages 2-23 through 2-27 of Unit 2. [Note 4]

Having learned linguistic devices which make it possible to state their discoveries about real numbers concisely it is natural for students to wonder how many such discoveries they need make. Is it possible that some can be justified, or even predicted, on the basis of others? It is easy to see that one can disprove a generalization by finding a counter-example; but, how can

one prove a generalization? If one equates the property of being provable with that of following logically from the basic principles, it turns out that a generalization is provable if one has a uniform method for showing that each of its instances follows from appropriate instances of the basic principles. [Pages 2-31, 32, 33; TC[2-31, 32]a, b, c, and d] Such a method for showing that instances of, for example:

$$\forall_x \; 3(x2) = 6x$$

follow from basic principles is illustrated in the case of the instance '$3(5 \cdot 2) = 6 \cdot 5$' by:

$3(5 \cdot 2) = 3(2 \cdot 5)$	[commutative principle for multiplication]
$3(2 \cdot 5) = (3 \cdot 2)5$	[associative principle for multiplication]
$(3 \cdot 2)5 = 6 \cdot 5$	[$3 \cdot 2 = 6$]

Hence, $3(5 \cdot 2) = 6 \cdot 5$.

(In Unit 2, the fact that $3 \cdot 2 = 6$ is accepted as a "computing fact"; the derivation of such computing facts from definitions, such as '$2 = 1 + 1$', and the basic principles is taken up in Unit 7.)

Using, say, 'a' as a real variable, the testing method illustrated above can be indicated by a test-pattern:

	$3(a \cdot 2) = 3(2 \cdot a)$	[cpm]
	$3(2 \cdot a) = (3 \cdot 2)a$	[apm]
	$(3 \cdot 2)a = 6a$	[$3 \cdot 2 = 6$]
Hence,	$3(a \cdot 2) = 6a.$	

The foregoing is conceived as a pattern which can be used to test any instance of '$\forall_x \; 3(x2) = 6x$'. For example, all that is needed to show that, say, '$3(7 \cdot 2) = 6 \cdot 7$'is a consequence of appropriate instances of the basic principles is to substitute '7' for 'a'. The test-pattern can be used to confound anyone who claims to have a counter-example to the generalization and, so, merits being admitted as a proof of the generalization. [Note 5]

If one compares the foregoing test-pattern with the "work" expected of a beginning student who is to "simplify" '$3(x \cdot 2)$' to '6x' the connection between simplifying algebraic expressions and proving elementary theorems about fields becomes evident. Of more immediate import is the fact that students can learn to use their knowledge of basic principles both to discover computational shortcuts and to discover errors in procedures (such as simplifying '1 + 2x' to '3x') which they have adopted in hope, but which have proved to lead to disaster.

In connection with simplifying expressions, the notion of equivalent expressions is introduced. [Pages 2-49, 50, 51; TC[2-51]a]

After considerable practice in simplifying algebraic expressions--both through strict adherence to the basic principles and by the free-wheeling methods which successful application of the former procedure suggest--students are brought back to something nearer mathematics than symbol-pushing

by a short discussion on theorems and basic principles and an opportunity to organize their knowledge of subtraction and division. [Pages 2-60, 61, and TC] In the remainder of Unit 2, students discover and prove a considerable number of theorems. [These are collected in the COMMENTARY for page 2-61.] Examples:

$$\forall_x \; x \cdot 0 = 0$$

$$\forall_x \forall_y \; \text{if } x + y = 0 \text{ then } -x = y$$

$$\forall_x \forall_y \; -(x - y) = y - x$$

$$\forall_x \forall_y \forall_z \; x(y - z) = xy - xz$$

$$\forall_x \forall_y \; \text{if } xy = 0 \text{ then } x = 0 \text{ or } y = 0$$

$$\forall_x \forall_{y \neq 0} \forall_u \forall_{v \neq 0} \; \frac{x}{y} + \frac{u}{v} = \frac{xv + uy}{yv}$$

$$\forall_x \forall_{y \neq 0} \; -\frac{x}{y} = \frac{-x}{y}$$

Interspersed with this "theoretical" development are numerous simplification exercises on which students can perfect the symbol-pushing techniques whose justification lies in the theorems they have proved.

*

Summarizing Units 1 and 2, it should be noted that students begin by becoming acquainted with the real numbers through comparing and contrasting the ways in which they and the more familiar numbers of arithmetic can be used in solving physical problems. On the basis of this acquaintance they accept some basic principles which describe the basic properties of the fundamental operations on real numbers. Having learned to use variables and quantifiers in order to give concise statements of these principles, they next learn how to derive theorems from them. Along the way, the procedures used in leading students to discover much of this theoretical structure for themselves give ample opportunity for them to develop the manipulating skills whose rationale is this basic structure and which are needed in furthering its development. At all times students feel that they are discussing a "real" subject matter--the real numbers--but their actual procedure is much like that of one who would abstract the notion of a field from examples like that furnished by the real number system, and, having done so, would proceed to develop the elementary theory of this kind of mathematical structure. This early introduction of proof pays off throughout the course, and particularly in the deductive development of Euclidean plane geometry in Unit 6.

*

Notes

1. A real variable is a symbol and is no more subject to change than is any other physical object. Hence the word 'variable' tends to be

misleading. On this subject I can't refrain from quoting from Professor E. J. McShane's Theory of Limits (MAA Film Manual, No. 2, p. 3). Discussing common misconceptions concerning the limit concept, Professor McShane writes:

> Incidentally, all these offside notions have one bad feature in common. They all involve the idea that b_j is "doing something". ... There is something alluring about the idea that j has a personality and "goes from 1 to 2 to 3" and so on, and that b_j "does" something like getting closer to 0. But this trick of personifying j and b_j is misleading even for sequences, and in more complex situations it is worse. ...
>
> In case you have been thinking of the b_j as a brisk little thing, jumping from b_1 to b_2 to b_3, and so on, don't blush too hard. Not so long ago that was a pretty customary way of thinking of it, and the custom died hard. ...

2. Here is an amusing example of a use of pronouns that strictly parallels the use of free variables in equation-solving problems:

> Identify the person described by the following sentences:
>
> He was a president of the United States.
> He commanded American armed forces.
> He has (had) a name consisting of six letters.
> He has (had) another name consisting of ten letters.
> He died in the nineteenth century.

The word 'he' is, here, a real variable whose domain is the set of all male human beings (living or not). The problem is to find such a person who satisfies (is a solution of) all five sentences.

3. This linking function of variables is well expounded by Quine in his Mathematical Logic (Cambridge, 1958), pp. 67-71. As Quine points out, the concept goes back at least to Peano's Formulaire of 1897, and was also exploited by Moses Schönfinkel in his "Über die Bausteine der Mathematischen Logik" (Math. Annalen, vol. 92 (1924), pp. 305-316). Schönfinkel's position is that

> ... the variable in a logical proposition serves only as a mark distinctive of certain argument-places and operators as mutually relevant. ...

4. The inadequacy of using real variables (instead of quantifiers and apparent variables) in stating generalizations should be apparent to anyone who has attempted to teach the distinction between, say, continuity at each point of a set and uniform continuity on that set. If the use of quantifiers and apparent variables turned out to be difficult for students to master, one might argue for reserving these concepts for a course in function theory. However, a great deal of experience shows that UICSM students have no particular difficulty with these concepts. Early familiarity with them should place those students who continue in mathematics

in a good position to appreciate the pointwise-uniform distinction. Actually, UICSM is more interested in the great majority of students who will not develop into mathematicians. Attention to linguistic matters such as the various roles played by variables appears not only to make it easier for such students to make sense out of mathematics, but also sharpens their appreciation for correct use of language, generally.

5. It should perhaps be noted that, while '$3(5 \cdot 2) = 3(2 \cdot 5)$' is not an instance of the cpm, it is a consequence of the instance '$5 \cdot 2 = 2 \cdot 5$'. This is the case because '=' refers to the logical relation of identity and, so, multiplication having been accepted as an operation, '$\forall_x \forall_y \forall_z$ if $x = y$ then $zx = zy$' is a tautology. --H. E. V.

* * *

NEWS AND NOTICES

The following publications and reprints of articles are now available upon request from the UICSM project office:

UICSM Information Sheet.

Max Beberman. "Improving High School Mathematics Teaching." Reprinted from Educational Leadership, December, 1959.

William Hale. "UICSM's Decade of Experimentation." Preprinted from The Mathematics Teacher.

Gertrude Hendrix. "Variable Paradox--A Dialog in One Act." Reprinted from School Science and Mathematics, June, 1959.

Gertrude Hendrix. "Learning by Discovery." Reprinted from The Mathematics Teacher, May, 1961.

M. Eleanor McCoy. "A Secondary School Mathamatics Program." Reprinted from The Bulletin of the National Association of Secondary School Principals, May, 1959.

UICSM Staff. "Words, 'Words', "Words"." Reprinted from The Mathematics Teacher, March, 1957.

UICSM Staff. "Arithmetic With Frames." Reprinted from The Mathematics Teacher, April, 1957.

Table of Contents, Units 1-8, HIGH SCHOOL MATHEMATICS.

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Correction: There is an error on page TC[7-134]d of the commentary for Unit 7. Line 4 should begin:

and only if ☆ is associative and has

*

Mr. Ralph Futrell of Catalina High School, Tucson, Arizona, reports that he and Mrs. Katherine Sassé of Pueblo High School in Tucson are conducting UICSM courses for teachers in their city. Mrs. Sassé is teaching First Course to 22 people, while Mr. Futrell has a group of ten studying Units 5 and 6. He notes that "There is a greater amount of interest in UICSM than ever before here in Tucson."

Mr. Howard Marston of the Principia School, St. Louis, Missouri, reviewed Unit 5, Relations and Functions, of HIGH SCHOOL MATHEMATICS in the October, 1961, issue of The Mathematics Teacher, pages 456 and 457.

Mr. Beberman's speeches and visits during September and October included the following:

September 15, 16	SMSG panel on tests, New York City.
September 20	Spoke to the Tazewell County, Illinois, branch of the American Association of University Women, at Pekin.
October 9	Long distance telephone speech, discussing the Illinois program for the district meetings of the Idaho Education Association.
October 9	Chicago Elementary Teachers Club, Chicago.
October 14	Panelist at annual meeting of the Illinois Council of Teachers of Mathematics, Urbana.
October 16, 17, 18	Osgood Hill Conference, Boston University, Andover, Mass.
October 19	Visited Matignon High School, Cambridge, Mass.
October 20	Central Western Zone of the New York State Teachers Association.
October 27	Northwestern Ohio Teachers Association, Toledo.

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